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Decay of Solutions of Systems of Hyperbolic Conservation Laws

James Glimm and Peter D. Lax

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Abstract

We are dealing with a system of two partial differential equations of the form

$$u_t + f_x = 0, \quad v_t + g_x = 0. \tag{1}$$

f and g are functions of u and v subject to the following conditions:

(a) The system (1) is hyperbolic.

(b) The system (1) is genuinely nonlinear in the sense of [4]

(c) The interaction of two shocks of the same family always produces a shock of the same family plus a rarefaction wave of the opposite family.

Denote by ε the oscillation of the initial data. The initial data need not have bounded total variation. Our results are as follows:

Theorem 5.1. For epsilon small enough the solution of the initial value problem exists for all positive time.

Theorem 5.2. If the initial data are periodic the solution decays like $1/t$.

The proofs are based on methods introduced in [2] supplemented by further inequalities which take into account the interaction of shock waves and rarefaction waves of the same family. The new estimates prove that such interactions must occur and that when they do occur they lead to decay in the solution.

The inequalities presented in this paper can also be used to prove if the initial data are constant outside of a compact interval, the solution decays like $1/(t)^{1/2}$.

DECAY OF SOLUTIONS OF SYSTEMS OF HYPERBOLIC CONSERVATION LAWS.[1]

by

James Glimm[2] and Peter D. Lax[3]

Preface

In this paper we study coupled systems of nonlinear conservation equations:

$$u_t + f_x = 0, \quad v_t + g_x = 0,$$

f, g functions of u and v. We assume that this system is strictly hyperbolic, i.e. has two real and distinct characteristic speeds ρ and χ. These characteristic speeds depend on u and v; it is well known that this causes in general the appearance of discontinuities, i.e. shock waves. Our aim is to show that the dependence of the characteristic speeds on u and v also causes shock waves and rarefaction waves of the same family to run into each other and to cancel each other out. Our main estimate, Theorem 4.1, is a quantitative form of this observation. This estimate is used to prove Theorem 5.1, the existence of solutions with arbitrarily prescribed bounded, measurable initial-data, with small oscillation. Another application is Theorem 5.2, according to which space-periodic solutions satisfy the decay law

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[1] The results of this paper were announced in [3].

[2] Supported in part by the National Science Foundation grant NSF GP 7477.

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$$|u-\bar{u}| \leq \frac{\text{const}}{t}, \quad |v-\bar{v}| \leq \frac{\text{const}}{t}$$

where $\bar{u}$, $\bar{v}$ denote the mean value of u and v per period.

Thus we find for these equations that a decay phenomenon for the solutions leads to a new estimate for the solution, an improved existence theorem, and a bound on the asymptotic behavior of the solutions for large times.

An important ingredient of our method is an analysis of the interaction of the two characteristic modes of propagation; we show that if the solution differs little from a constant, the interaction is negligible. Since this result is used essentially in our estimates, we can only deal with weak signals, i.e. solutions whose initial state differ little from a constant.

We give now an intuitive description of the ideas involved illustrating them first on a single conservation equation

$$u_t + f_x = 0. \tag{P.1}$$

The differentiated form of the equation is

$$u_t + f'(u)u_x = 0.$$

We take for f the simplest nonlinear function:

$$f(u) = \frac{a}{2} u^2, \quad a \text{ positive constant.}$$

The nonconservation form of (P.1) is then

$$u_t + auu_x = 0, \tag{P.2}$$

which shows that u is constant along curves satisfying

$$\frac{dx}{dt} = au. \tag{P.3}$$

Since u is constant along them, these curves, called characteristics, are straight lines. In general these straight lines intersect; at a point of intersection the solution would be multivalued.

Instead of becoming multivalued the solution becomes discontinuous, satisfying equation (P.1) in the sense of distributions. For piecewise continuous solutions this means the fulfillment of the jump relation

$$\frac{dx}{dt} = \frac{[f]}{[u]} = s \tag{P.4}$$

where $x = x(t)$ is the curve of discontinuity, and

$$[f] = f_r - f_\ell \quad , \quad [u] = u_r - u_\ell$$

is the difference between the values of f, respectively u, on the right and left side of $x(t)$. For our special choice of f (P.4) becomes

$$\frac{dx}{dt} = a \; \frac{u_\ell + u_r}{2} \; . \tag{P.5}$$

It is well known that only those discontinuous solutions are acceptable, either mathematically or physically, which satisfy the so-called entropy condition, which requires that

(P.6) $$f'(u_\ell) > \frac{[f]}{[u]} > f'(u_r).$$

Geometrically this condition asserts that the characteristics on either side of the discontinuity, when drawn in the direction of increasing t, run into the curve of discontinuity:

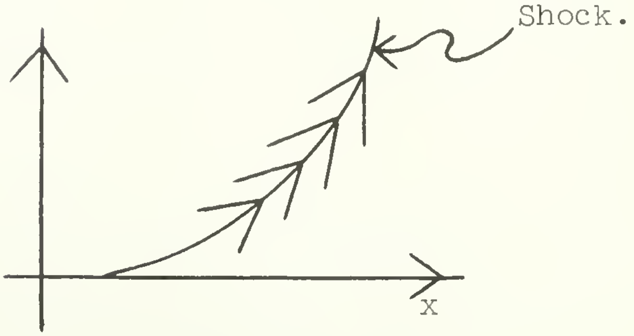

Fig. P.1

Discontinuities satisfying (P.6) are called shocks.

Note that for our choice of $f = \frac{1}{2} au^2$, the entropy condition takes the form

$$au_\ell > a\frac{u_\ell+u_r}{2} > au_r,$$

which for $a > 0$ is equivalent with

(P.7) $$u_\ell > u_r.$$

We return now to continuous solutions; let $x_1(t)$ and $x_2(t)$ be a pair of characteristics, $0 \leq t \leq T$; it follows from the description of u as constant along characteristics that $u(x,0)$ on the interval $[x_1(0),x_2(0)]$ and $u(x,T)$ on the interval $[x_1(T), x_2(T)]$ are equivariant. In particular *the increasing and decreasing variation respectively of* u *on these two intervals are the same*.

More generally, let σ_1 and σ_2 be any pair of curves connecting $x_1(t)$ and $x_2(t)$, neither of them tangent to a characteristic; then likewise u on σ_1 is equivariant with u on σ_2.

Suppose that there are shocks present; let $y_1(t)$ and $y_2(t)$ be a pair of characteristics which run into shocks at times t_1 and t_2, respectively. Define $y_i(t)$ for $t > t_i$ as the shock into which y_i has run. Let T be some value $> t_i$; there are two points x_1 and x_2 in the interval $[y_1(0), y_2(0)]$ such that the characteristics issuing from x_i intersects the shock y_i at time T:

$$x_i(T) = y_i(T).$$

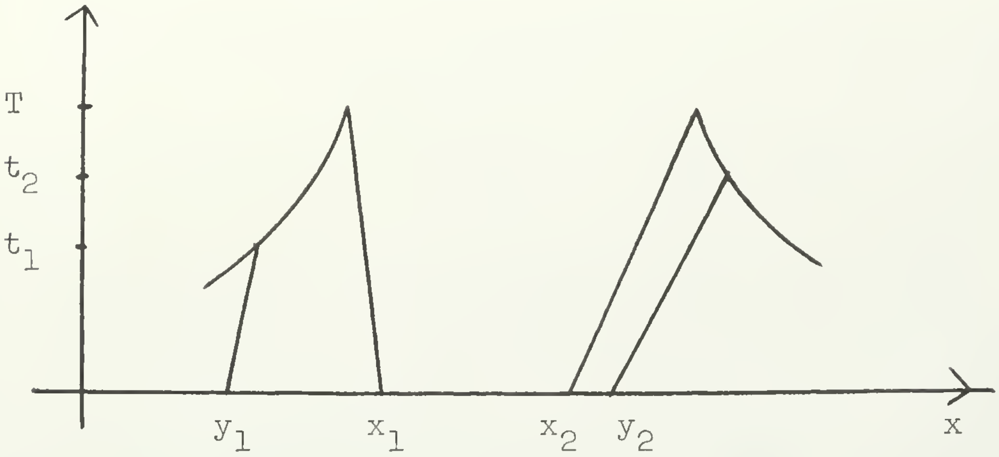

Fig. P.2

If there are no further shocks contained between x_1 and x_2 for $0 \leq t \leq T$ then $u(x,0)$ on the open intervals $(x_1(0), x_2(0))$ and $u(x,T)$ on the open interval $(x_1(T),x_2(T))$ are equivariant ; otherwise $u(x,T)$ has a smaller variation. This shows that the effect of characteristics running into shocks is to decrease the total variation $u(x,t)$. We make now a quantitative estimate of this decrease.

Denote by $D(t)$ the width of the strip bounded by $x_1(t)$ and $x_2(t)$:

$$D(t) = x_2(t) - x_1(t).$$

Differentiating with respect to t we get, upon using relation (P.3), that

$$\frac{d}{dt} D(t) = \frac{d}{dt} x_2 - \frac{d}{dt} x_1 = au_2 - au_1.$$

Since u_2 and u_1 are constant along characteristics, we deduce after integrating with respect to t that

$$\text{(P.8)} \qquad D(T) = D(0) + a(u_2-u_1)T.$$

Let L_o be any interval on the x-axis; we subdivide it into subintervals $[y_{j-1},y_j]$, $j=1,\ldots n$, on which $u(x,0)$ is monotonic, increasing for j odd, decreasing for j even. As before we define $y_j(t)$ as the characteristic issuing from y_j, continued as a shock if it runs into a shock. Denote by $L(t)$ the interval $[y_o(t),y_n(t)]$.

The analysis presented before shows that for all t

$u(x,t)$ is increasing on $(y_{j-1}(t), y_j(t))$ for j odd, decreasing for j even; according to (P.7) u decreases across shocks. Therefore the total increasing variation of $u(x,t)$ across $L(t)$ is

$$\sum_{j \text{ odd}} u_j(t) - u_{j-1}(t), \tag{P.9}$$

where $u_{j-1}(t)$ denotes the value of u on the right edge of $y_{j-1}(t)$, $u_j(t)$ the value of u on the left edge of $y_j(t)$.

Denote as before by $x_{j-1}(t)$, $x_j(t)$ characteristic starting inside y_{j-1}, y_j which intersect $y_{j-1}(t)$, respectively $y_j(t)$ at $t = T$. $u_j(t)$ is then the constant value of u on $x_j(t)$. Denoting $x_j(t) - x_{j-1}(t)$ by $D_j(t)$ we obtain from (P.8) that

$$D_j(T) = D_j(0) + a[u_j(t) - u_{j-1}(t)]T$$

Summing over odd j we get

$$\sum D_j(T) = \sum D_j(0) + aT \sum u_j - u_{j-1}.$$

The intervals $[x_j, x_{j-1}]$ are disjoint and lie in $L(t)$; therefore the sum of their length cannot exceed the length $|L(T)|$. Using this and (P.9) we can turn (P.10) into the following inequality:

$$\text{Total increasing variation of } u(x,T) \text{ over } L(T) \le \frac{|L(T)|}{aT}. \tag{P.11}$$

Suppose in particular that $u(x,t)$ is periodic in x with period p; take L_o to be of length p; then by periodicity $L(t)$ has length p for all t and we deduce from (P.11) that

$$\text{(P.12)} \qquad \frac{\text{Total variation of } u(x,t) \text{ per period}}{p} \leq \frac{1}{aT} ,$$

i.e. that periodic solutions decay like $1/T$. It is remarkable that the bound on the right side is independent of the particular solution considered.

We wish to generalize our method to systems of two equations. It is well known that there exist *Riemann invariants* w and z which are functions of u and v and which satisfy equations of the form

$$w_t + \rho w_x = 0, \quad z_t + \lambda z_x = 0,$$

where ρ and $\hat{z}$ are functions of w and z. These equations express the *constancy* of w, respectively z, along curves $x(t)$ satisfying

$$\frac{dx}{dt} = \rho \ , \quad \text{respectively} \quad \frac{dx}{dt} = \lambda.$$

These curves are called ρ-characteristics and λ-characteristics. It follows as before that if $x_1(t)$, $x_2(t)$ are two ρ-characteristics, then $u(x,0)$ along $[x_1(0),x_2(0)]$ and $u(x,T)$ along $[x_1(T),x_2(T)]$ are equivariant.

The ρ-characteristics are no longer straight lines; nevertheless it can be shown that in general they intersect, provided that ρ genuinely depends on w, i.e. that $\rho_w \neq 0$. This leads to the formation of shock waves. The analogue of condition (P.6) for systems is

(P.13) $$\rho(u_\ell,v_\ell) > s > \rho(u_r,v_r),$$

where s is the speed of propagation of the discontinuity. Discontinuities satisfying (P.13) are called ρ-shocks; similarly there are λ-shocks. The geometric interpretation of (P.13) is that ρ-characteristics drawn in the direction of increasing t run into ρ-shocks; from this we deduce as before that the presence of ρ-shocks decreases the total variation of w.

To proceed further we have to estimate also the effect of λ-shocks on the total variation of w. Here it is important to recall that across weak λ-shocks the change in w is proportional to (change in $z)^3$; this is basic to the following estimate

(P.14) Change in total variation of w due to λ-shocks

$$\leq O(\varepsilon)V^2$$

where ε is the oscillation of the solution, V its initial total variation.

There is a second difficulty we have to cope with:

The differential equation for the width of a strip bounded by ρ characteristic, $D(t) = x_2(t) - x_1(t)$, is as follows:

$$\frac{d}{dt} D = \rho_2 - \rho_1 = \overline{\rho}_w(w_2-w_1) + \overline{\rho}_z(z_2-z_1),$$

where on the right we have applied the mean value theorem. Since we are dealing with solutions whose total oscillation ε is small, $\overline{\rho}_w$ and $\overline{\rho}_z$ themselves oscillate $O(\varepsilon)$. The quantities w_1 and w_2 are independent of t, but z_1 and z_2 are not! We can overcome this difficulty by measuring the width of the strip bounded by ρ-characteristics not between points with the same t coordinates but between points which lie approximately on the same λ-characteristics. Since z is constant along λ-characteristics this makes the quantity $z_2 - z_1$ appearing in the differential equation small.

In Section 1 we carry out the details of estimating $D(T)$ for solutions without shocks.

In Section 2 we construct approximate characteristics for difference approximations introduced in [2]. We derive approximate conservation laws of the increasing and decreasing variation of w and z; these laws are formulated as a balance between the amount of shock wave and rarefaction wave of either kind entering and leaving a region, the amount of rarefaction and shock wave of the same family cancelling each other in the region, and a correction term accounting for the interaction between waves belonging to different families.

In Section 3 we construct by passage to the limit characteristic curves and shock curves for exact solutions, and derive inequalities for the variations of w and z.

In Section 4 we study the spreading of strips bounded by characteristics in the presence of shocks and deduce an analogue of inequality (P.11).

In Section 5 we use this inequality to construct solutions whose initial values are not of bounded total variation. For periodic solutions we deduce the analogue of (P.12).

A word of caution: the proof of lemma 3.4 is heavy going and we advise the reader to read the rest of this paper first.

As we have indicated, we can only deal with solutions whose oscillation is small. Recently, [5], Nishida succeeded in solving the initial value problem in the large for the special system: $f = v$, $g = -a/u$. It would be interesting to study the decay of solutions of the system considered by Nishida.

1. The Widening of Rarefaction Waves.

In this section we carry out rigorously the idea alluded to at the end of the preface: we obtain a lower bound for the distance of two characteristics in terms of the amount of rarefaction wave contained between them; here we only treat shockless solutions.

The equations under consideration are of the form

$$u_t + f_x = 0, \quad v_t + g_x = 0 \tag{1.1}$$

where f and g are nonlinear functions of u and v. (1.1) can be written as a quasilinear system

$$\begin{aligned} u_t + f_u\, u_x + f_v\, v_x &= 0, \\ v_t + g_u\, u_x + g_v\, v_x &= 0. \end{aligned} \tag{1.2}$$

We assume that (1.2) is hyperbolic, i.e. that the coefficient matrix of the x-derivatives,

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$$

has real and distinct eigenvalues in some neighborhood of a point which we take to be the origin (0,0). It is well known that for any quasilinear hyperbolic system(1.2)for two functions Riemann invariants can be introduced as new variables; call these w and z. In terms of these the differential equations (1.2) take on the simple form

$$\begin{aligned} w_t + \rho\, w_x &= 0 \\ z_t + \lambda\, z_x &= 0 \end{aligned} \tag{1.3}$$

where ρ and λ are functions of z and w. We assume that ρ and λ genuinely depend on w and z respectively, i.e. that ρ_w and λ_z are nonzero. Replacing w by $-w$, z by $-z$ if necessary these conditions of genuine nonlinearity can be put in the form

$$\rho_w > 0, \quad \lambda_z > 0. \tag{1.4}$$

λ and ρ are called characteristic speeds; curves $x(t)$ propagating with characteristic speed, i.e. satisfying

$$\frac{dx}{dt} = \lambda \quad , \quad \frac{dx}{dt} = \rho \tag{1.5}$$

are called λ-, respectively ρ-characteristics. Since λ and ρ vary from solution to solution, so do characteristic curves.

We assume that λ and ρ are distinct, say $\lambda < \rho$. Since in what follows w and z differ little from (0,0), we may without loss of generality take $\lambda < 0 < \rho$; then all λ-characteristics move to the left, ρ-characteristics to the right; this contributes to the clarity of diagrams.

Equation (1.3) states that w is constant along ρ-characteristics, z constant along λ-characteristics. This fact is basic to all that follows; we shall use it to prove the following

Theorem 1.1: *Suppose that* λ *and* ρ *are twice differentiable functions of* w *and* z. *Let* $z(x,t)$, $w(x,t)$ *be a differentiable solution of* (1.3) *which is periodic in* x *with period* p *and which exists for* $0 \leq t \leq T$. *Then*.

$$T \leq \frac{\text{const } p}{\omega + \zeta} + p$$

where

ω = total variation per period of $w(x,0)$

ζ = total variation " " " $z(x,0)$

provided that

$$\varepsilon = \text{Max}\left\{|w(x,0)|, |z(x,0)|\right\}$$

is less than some constant.

Remark: Since solutions are uniquely determined by their initial data it suffices to assume that the initial values of w and z are periodic.

Proof: From the constancy of Riemann invariants along characteristics it follows from the definition of ε that

(1.6) $$|z(x,t)| < \varepsilon, \quad |w(x,t)| < \varepsilon$$

for all x and all t.

Let x_1 and x_2 be any two points, say $x_1 < x_2$, denote by $x_1(t)$ and $x_2(t)$ the ρ-characteristics with respect to the solution under consideration issuing from these points:

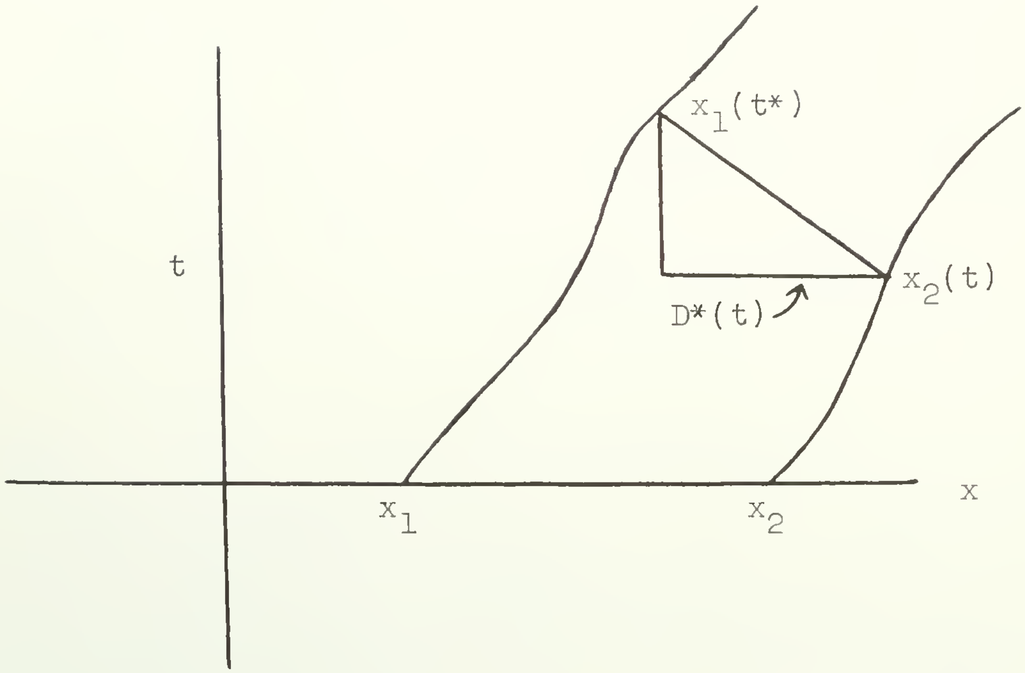

Fig. 1.1

Denote $\lambda(0,0)$ by λ_o; we draw a spacelike straight line propagating with a speed λ_1 through the point $(x_2(t),t)$ and denote by $t^*(t)$ the time at which this line intersects the curve $x_1(t)$, see Fig. 1. t^* satisfies

$$x_2(t) - x_1(t^*) = \lambda_1(t-t^*). \tag{1.7}$$

Differentiating with respect to t we get

$$\rho_2 - \rho_1\dot{t}^* = \lambda_1(1-\dot{t}^*) \tag{1.8}$$

where $\cdot$ denotes differentiation with respect to t and ρ_1 and ρ_2 abbreviate $\dot{x}_1$ and $\dot{x}_2$. Solving for $\dot{t}^*$ we get

$$(1.9) \qquad \dot{t}^* = \frac{\rho_2 - \lambda_1}{\rho_1 - \lambda_1} = 1 + \frac{\rho_2 - \rho_1}{\rho_1 - \lambda_1} .$$

We denote by $D^*(t)$ the horizontal width of the strip between $x_1(t^*)$ and $x_2(t)$ in the λ_1 direction, i.e.

$$(1.10) \qquad D^*(t) = x_2(t) - x_1(t^*).$$

Differentiating (1.10) with respect to t we get, using (1.9), that

$$(1.11) \qquad \frac{d}{dt} D^* = \rho_2 - \rho_1 \dot{t}^* = (\rho_2 - \rho_1) \frac{\lambda_1}{\lambda_1 = \rho_1} = [\rho] \frac{\lambda_1}{\lambda_1 - \rho_1}$$

Here $[\rho]$ abbreviates $\rho_2(t) - \rho_1(t^*)$.

We choose now

$$\lambda_1 = \lambda(0,0) - \text{const } \varepsilon,$$

the constant chosen so large that the line with slope λ_1 is spacelike. This is possible since $|w|$ and $|z|$ never exceed ε; for the same reason, $\rho_1 = \rho(0,0) + O(\varepsilon)$. Using the abbreviations

$$\rho(0,0) = \rho_o, \quad \lambda(0,0) = \lambda_o$$

we have therefore

$$\frac{\lambda_1}{\lambda_1 - \rho_1} = \frac{\lambda_o}{\lambda_o - \rho_o} (1 + O(\varepsilon));$$

so (1.11) can be rewritten as

$$(1.12) \qquad \frac{d}{dt} D^* = [\rho] \frac{\lambda_o}{\lambda_o - \rho_o} (1 + O(\varepsilon)).$$

Using the mean value theorem and the abbreviations

$$[w] = w(x_2(t),t) - w(x_1(t^*),t^*)$$

(1.13)

$$[z] = z(x_2(t),t) - z(x_1(t^*),t^*)$$

we get

$$[\rho] = a[w] + b[z]$$

where $a = \rho_w$, $b = \rho_z$ at some intermediate points. We substitute this into (1.12); we can write the result as

(1.14) $$\frac{d}{dt} D^* = c_o[w](1+O(\varepsilon)) + g[z]$$

where

$$c_o = a_o \frac{\lambda_1}{\lambda_o-\rho_o} , \quad g = b \frac{\lambda_o}{\lambda_1-\rho_1} = b \frac{\lambda_o}{\lambda_o-\rho_1} (1+O(\varepsilon)).$$

Since according to (1.4) ρ_w is positive, so is c_o.

Next we estimate [z]; for this we make use of the constancy of z along λ-characteristics:

(1.15) $$z(x_1(t^*),t^*) = z(x_2(t-t_o),t-t_o)$$

where $t - t_o$ is the time when the backward λ-characteristic through $x_1(t^*),t^*$ intersects the curve $x_2(t)$, as shown on figure 1.2 below:

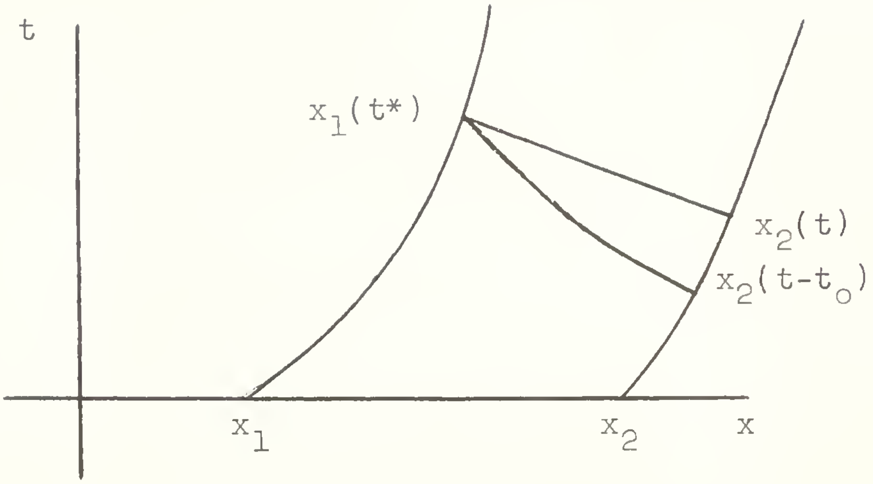

Fig. 1.2

Since w and z oscillate at most ε, the speed λ differs at most by $O(\varepsilon)$ from λ, chosen earlier. The time difference t_o is bounded by the difference of the reciprocals of the speed times the distance to be travelled:

$$t_o(t) = O(\varepsilon)D^*(t). \tag{1.16}$$

Using (1.13) and (1.15) we can write

$$\begin{aligned}[z] &= z(x_2(t),t) - z(x_1(t^*),t^*) \\ &= z(x_2(t),t) - z(x_2(t-t_o,t-t\ ) \\ &= \int_{t-t_o}^{t} \frac{d}{ds}\, z(x_2(s),s)ds\end{aligned}$$

Substituting this into (1.14) gives

$$\frac{d}{dt} D^*(t) = c[w] + g \int_{t-t_o}^{t} z_s(s)ds$$

where

$$z_s = \frac{d}{ds} z(x_2(s),s).$$

and

$$c = c_o(1+O(\varepsilon)).$$

Since w is constant along both characteristics $x_1(t)$ and $x_2(t)$ it follows that so is $[w]$. Therefore integrating the above with respect to t from 0 to T gives

$$D^*(T) = D^*(0) + c[w]T + \int_0^T \int_{t-t_o}^{t} |gz_s(s)|dsdt. \tag{1.17}$$

To estimate the last term on the right, we interchange the order of integration

$$\int_0^T \int_{t-t_o}^{t} |g(z_s)|(s)dsdt = \int_0^T \int_s^{s+t_1(s)} |g||z_s|dt\, ds + q \tag{1.18}$$

where t_1 is defined in figure 3 below:

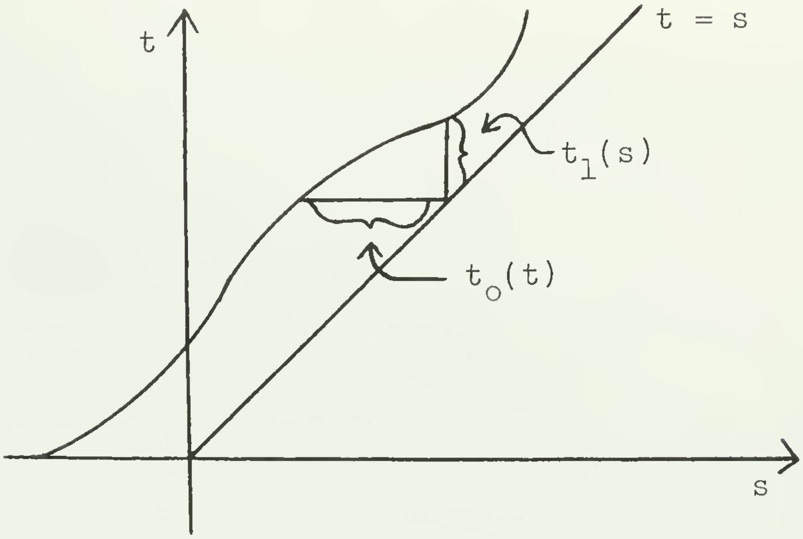

Fig. 1.3

and q denotes the integral over a triangular piece:

(1.19) $$q = \int_{-t_o(0)}^{0} \int_{0}^{s+t_1(s)} |gz_s| dt\, ds.$$

To estimate the right side of (1.18) we need an estimate for t_1; from figure 1.3 we get the relation

$$0 \leq t_1(s) = t_o(s+t_1(s)).$$

Using (1.16) we get

$$t_1(s) = O(\varepsilon)D^*(s+t_1(s)).$$

Since D^* is Lipschitz continuous

$$D^*(s+t_1(s)) = D^*(s) + O(t_1(s));$$

this combined with the above estimate for t_1 gives

$$t_1(s) = O(\varepsilon)D^*(s).$$

The function g is uniformly bounded; therefore using the above estimate for t_1 gives the following upper bound for the double integral on the right side of (1.18):

$$O(\varepsilon)\int_0^T |z_s(s)|D^*(s)ds.$$

We can estimate q in (1.19) similarly:

(1.20) $$|q| \leq O(\varepsilon)D^*(0)V$$

where

$$V = \int_{-t_o(0)}^{0} |z_s| ds \tag{1.21}$$

Combining (1.17) and (1.18) and using the above estimates we get the following two-sided estimate for $D^*(T)$:

$$\begin{aligned}[1-O(\varepsilon)V)][D^*(0) + c[w]T] - O(\varepsilon)\int_0^T |z_s(s)|D^*(s)|ds \leq D^*(T) \\ \leq [1+O(\varepsilon)V)][D^*(0) + c[w]T] + O(\varepsilon)\int_0^T |z_s(s)|D^*(s)ds.\end{aligned} \tag{1.22}$$

Next we make use of the following classical inequality:

Lemma: Let m be an increasing function and suppose that for each positive T, M satisfies the inequality

$$M(T) \leq F(T) + \int_0^T M(t)dm(t) \tag{1.23}$$

Then M satisfies the inequality

$$M(T) \leq M_1(T) \tag{1.24}$$

where M_1 is the solution of the integral equation obtained from (1.23) by replacing the sign of inequality by equality:

$$M_1(T) = F(T) + \int_0^T M_1(t)dm(t). \tag*{(1.23)$_1$}$$

The explicit expression for M_1 is

$$M_1(T) = e^{m(T)}F(0) + \int_0^T e^{m(T)-m(t)} \frac{dF}{dt} dt; \tag{1.25}$$

here $m(0)$ is taken to be 0.

Suppose that in addition to (1.23) M satisfies an inequality in the opposition direction:

$$(1.26) \qquad G(T) - \int_0^T M(t)dm \leq M(T).$$

Substituting the upper bound (1.24) into the left side of (1.26) and using $(1.23)_1$ we obtain the following lower bound for M:

$$(1.27) \qquad G(T) + F(T) - M_1(T) \leq M(T).$$

Suppose $F(t)$ is an increasing function; then the integral on the right in (1.25) will increase if the exponent $m(T) - m(t)$ is replaced by the larger quantity $m(T)$, and we get the following upper bound for M_1:

$$M_1(T) \leq e^{m(T)}F(T)$$

Substituting this into (1.27) gives

$$(1.28) \qquad G(T) + (1-e^{m(T)})F(T) \leq M(T).$$

We apply this to the function D^* as M; inequality (1.22) is inequalities (1.23) and (1.26), with

$$(1.29) \qquad m(t) = O(\varepsilon)\int_0^t |dz(s)| = O(\varepsilon)Z(t)$$

$$F(T) = [1+O(\varepsilon)V][D^*(0) + c[w]T]$$

$$G(T) = [1-O(\varepsilon)V][D^*(0) + c[w]T].$$

Inequality (1.28) in this case gives

$$(1.30) \qquad [2-e^{O(\varepsilon)Z(T)} - O(\varepsilon)V][D^*(0)+c[w]T] \leq D^*(T)$$

where $c = [1-O(\varepsilon)]c_0$.

The quantity $Z(t)$ defined in (1.29) is the total variation of z along the characteristic $x_2(t)$. Using again the constancy of z along λ-characteristics we deduce the following *law of conservation of total variation*:

Let C_1 *and* C_2 *be two curves, both transversal to* λ-*characteristics, whose left endpoints and right endpoints respectively lie on the same* λ-*characteristic* (see figure 1.4):

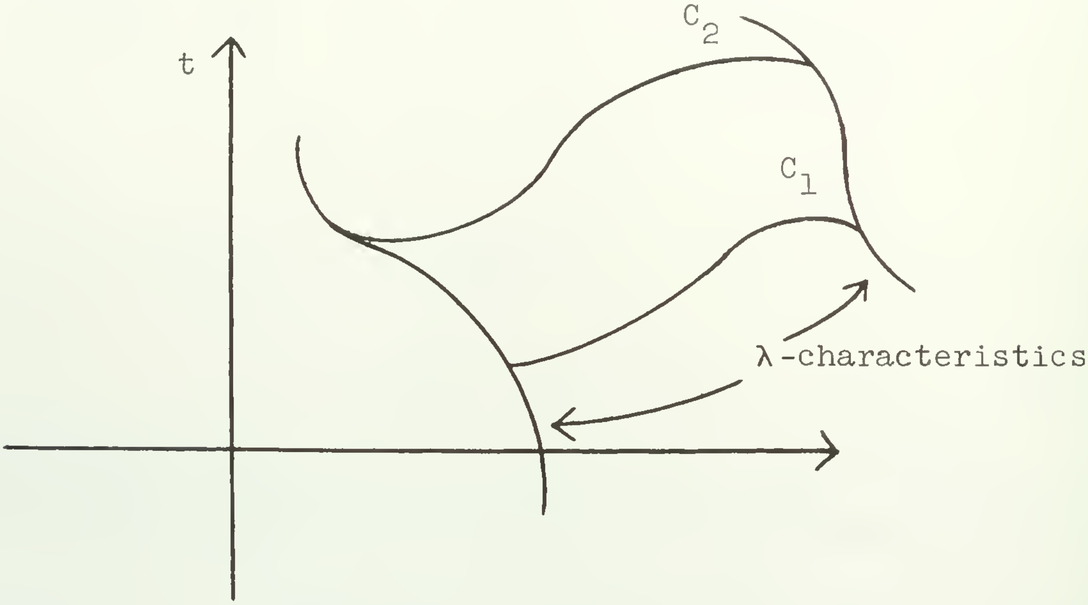

Fig. 1.4

Then z *along* C_1 *and* C_2 *are equivariant*.

It follows from this that the *total variation* of z along C_1 equals its total variation along C_2.

We apply this result to the case when C_2 is the

ρ-characteristic x_2 between $t = 0$ and $t = T$, and C_1 is the interval of the x-axis between x_2 and $y(T)$ defined as that point where the λ-characteristic through the point $x_2(T),T$ intersects the x axis, see figure 1.5:

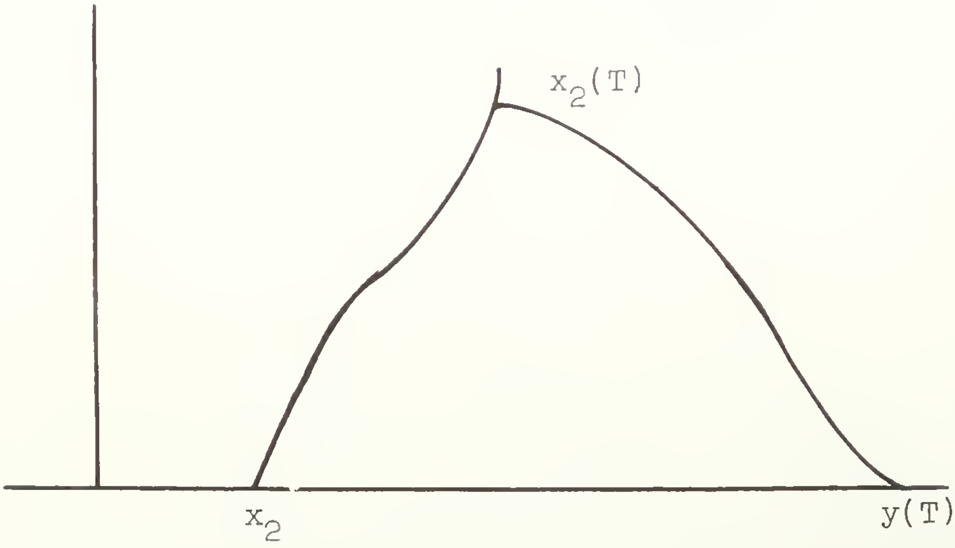

Fig. 1.5

Clearly the length of the interval $[x_2,y(T)]$ is $O(T)$.

We recall now that the total variation of the initial value of z per period is denoted by ζ; therefore the total variation of z along an interval of length L is $L\zeta/p$. Thus for $T > O(p)$

$$Z(T) = \text{Total variation of } z \text{ along } x_2(t) =$$

$$\text{total variation of } z \text{ in } (x_2,y) = O(\zeta T)/p.$$

A similar estimate for the quantity V defined in (1.21) gives $V \leq \zeta$ provided x_1 and x_2 lie within a single period. Using these estimates for Z and V in inequality (1.30) gives

$$(1.31) \qquad [2-e^{O(\varepsilon\zeta T)/p}-O(\varepsilon)\zeta]c[w]T \leq D^*(T).$$

We restrict ε, ζ and T so that this inequality is non-trivial, i.e. that the coefficient on the left is positive; this will be the case if the exponent $O(\varepsilon\zeta T)/p$ is less than, say, log 1.3, and $O(\varepsilon)\zeta$ is less than .2:

$$(1.32) \qquad T \leq \frac{\text{const}}{\varepsilon\zeta} p \quad , \quad \varepsilon\zeta \leq \text{const}.$$

For T, ε, ζ satisfying (1.32) inequality (1.31) implies

$$(1.33) \qquad c[w]T \leq 2D^*(T).$$

So far x_1 and x_2 were any two points within one period. We choose now a set of intervals $I^j = (x_1^{(j)}, x_2^{(j)})$ all within one period, say $(0,p)$, so that

i) the quantities $[w]^j = w(x_2^{(j)},0) - w(x_1^{(j)},0)$

are all positive

ii) $\omega/3 \leq \sum [w]^j$

where, as before, ω denotes the total variation of w per period.

Such a choice of intervals is possible since the variation of any function in the positive direction is exactly one half its total variation.

The points x_1^j, x_2^j were chosen to lie in an interval of length p. Consider the ρ-characteristic curves $x_1^j(t)$ and $x_2^j(t)$ issuing from these points. Since the solution under consideration is periodic with period p, it follows that for any t the points $x_1^j(t)$ and $x_2^j(t)$ also lie in an interval of length p. Denote by $D_j(t)$ the width of the strip bounded by the j^{th} pair of characteristics:

$$D_j(t) = x_2^j(t) - x_1^j(t).$$

It follows from the above analysis that

$$\sum D_j(t) < p.$$

We recall now the quantity $D_j^*(t)$ which measures the width of the j^{th} strip in the λ_1 direction. Clearly

$$D_j^*(t) \leq \text{const } D_j(t)$$

so summing we get from the above inequality that

$$\sum D_j^*(t) \leq \text{const } p.$$

On the other hand if T satisfies (1.32) then inequality (1.33) holds for each strip:

$$c[w]^j T \leq 2D_j^*(T) .$$

Summing with respect to j and using the facts that $\sum D_j^*(t) \leq \text{const } p$ and that $\sum [w]^j > \omega/3$ we get that

(1.34) $$\omega T \leq \text{const } p$$

provided that (1.32) is satisfied. Dividing by ω we get

$$(1.35) \qquad T \leq \frac{\text{const } p}{\omega} .$$

Since the roles of w and z are interchangeable, we may assume that $\xi \leq \omega$. In this case for ε less than a constant independent of ξ and ω the right side of (1.35) is less than (1.32); this contradicts our assertion that (1.32) implies (1.35). The resolution of this contradiction is that the solution does not exist for values of T violating (1.35). This completes the proof of Theorem 1.1.

2. Approximate conservation laws and approximate characteristics.

We start by outlining briefly the theory of shock waves, rarefaction waves and their interactions. For details see [4] and [2].

A discontinuous function which is a solution in the sense of distributions of the conservation laws (1.1) satisfies along the curve of discontinuity the jump relations

$$s[u] = [f], \; s[v] = [g] \tag{2.1}$$

where $[u]$ denotes the difference of the value of u on the right and on the left side of the discontinuity, and s denotes the speed with which the discontinuity propagates. It is well known that physically meaningful discontinuous solutions satisfy in addition the entropy condition; under the normalization (1.4) this takes the form

$$[w] < 0 \tag*{(2.2)$_r$}$$

for discontinuities propagating to the right (s positive),

$$[z] < 0 \tag*{(2.2)$_\ell$}$$

for discontinuities propagating to the left. Discontinuous solutions satisfying (2.1) and (2.2) are called shocks.

It has been shown in [4] that for weak shocks propagating to the right ($0 < s$)

$$[z] = O[w]^3 \tag{2.3}$$

while for weak shocks propagating to the left ($s < 0$)

$$[w] = O[z]^3.$$

For the equations of gas dynamics both z and w decrease across shocks propagating in either direction:

$$(2.4) \qquad [z] < 0, \quad [w] < 0 \quad \text{across shocks.}$$

We stipulate that this condition be fulfilled for the systems studied in this paper.

It follows from (2.1) and (2.2) that for weak shocks ($[z]$, $[w]$ small) the characteristic speed in the direction in which the shock is moving is greater than the shock speed behind the shock and less in front of it. I.e.,

$$(2.5) \qquad \begin{aligned} \rho_r < s < \rho_\ell \quad &\text{when} \quad 0 < s \\ \lambda_r < s < \lambda_\ell \quad &\text{when} \quad s < 0; \end{aligned}$$

here the subscript ℓ and r refer to the state to the left and to the right of the shock.

The simplest and most basic initial value problem is the Riemann problem where the initial value has one constant value a for x negative and another constant value b for x positive. It is not hard to show that, at least when a and b are close enough, this initial value problem has a solution, depending only on x/t; this solution may contain a shock propagating in either or both directions. The solution is constant in certain sectors and in those sectors where it isn't, either z or w is constant. These regions are called centered rarefaction waves; they propagate either to the right or to the left, and their boundaries are characteristic lines. Figure 2.1 presents one of the four possibilities:

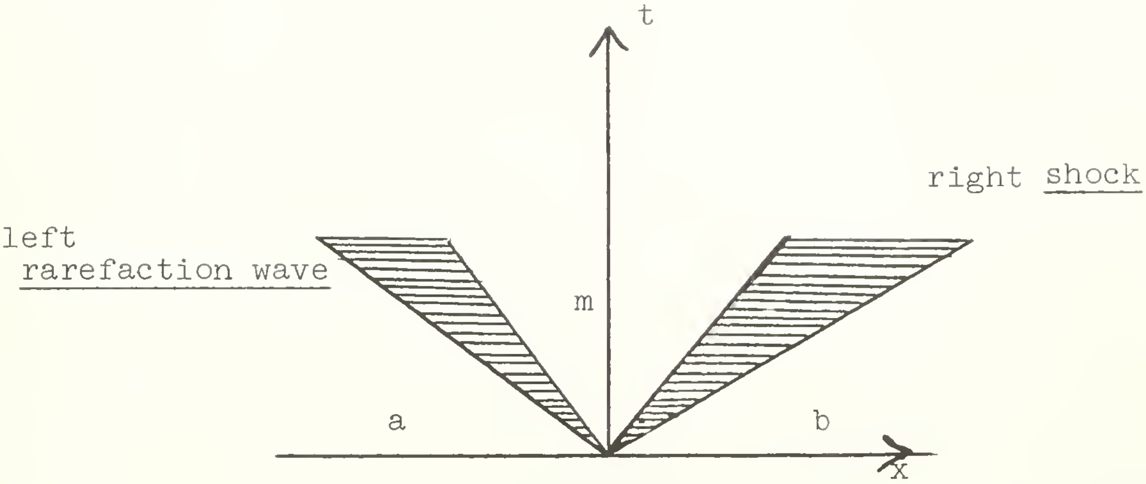

Fig. 2.1

Since the right edge of a centered rarefaction wave moves faster than the left edge, $[\rho]$ and $[\lambda]$ are positive for right, respectively left, rarefaction waves, where $[\quad]$ denotes the difference of the value of the function in brackets on the right edge of the rarefaction wave and its value on the left edge. In view of the normalization given in (1.4) these relations are equivalent with

$$0 < [w], \quad 0 < [z] \tag{2.6}$$

respectively.

We indicate on figure 2.2 the relative positions in the w, z plane of the constant states a, m and b

occurring in the interaction pictured in figure 2.1:

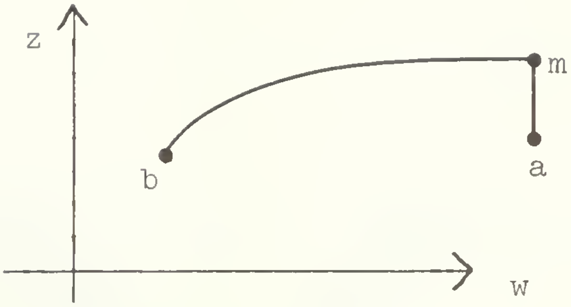

Fig. 2.2

Relations $(2.2)_\ell$, $(2.3)_\ell$ and $(2.4)_\ell$ for the left shock connecting a and m express the fact that m lies below a and slightly to the left of a, while the constancy of z in a rarefaction wave moving to the right and relation $(2.6)_r$ express the fact that b and m lie on the same horizonal line, b to the right of m.

Figure 2.3 shows two shock waves moving to the right which coalesce to form another shock wave moving to the right and a rarefaction wave moving to the left:

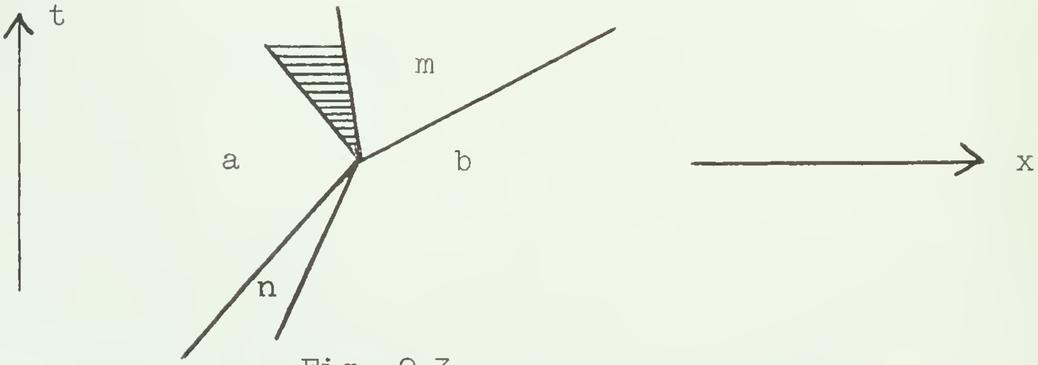

Fig. 2.3

The explanation of this phenomenon lies in relations (2.2), (2.3) and (2.4), as illustrated by figure 2.4:

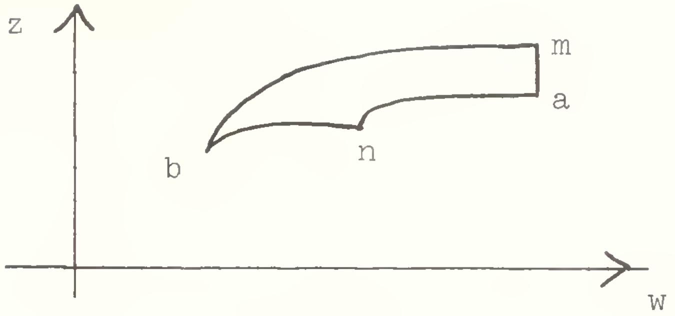

Fig. 2.4

We shall denote by $[z]$ and $[w]$ the magnitude of a wave propagating to the right and left respectively; according to (2.2) and (2.6) the magnitude of a shock wave is negative, of a rarefaction wave positive.

We shall denote by $(a,b)_\ell$ and $(a,b)_r$ the magnitude of the left and right waves which arise in the resolution of the Riemann problem with state a on the negative axis, state b on the positive axis. We shall say that a can be joined to b through the waves $(a,b)_\ell$ and $(a,b)_r$.

Suppose that a can be joined to b through a single rarefaction wave moving to the left; let c be another state which can be joined to b through a rarefaction wave moving to the left and some other wave moving to the right. Then clearly a can be joined to c through a left rarefaction wave which is the sum of those joining a to b and b to c, and the right wave $(b,c)_r$.

An important generalization of this simple observation **is given in a** the following theorem of [2]:

Theorem: Let a, b, c be any three nearby states; then

$$(a,c)_\ell = (a,b)_\ell + (b,c)_\ell + \text{error} \qquad (2.7)_\ell$$

and

$$(a,c)_r = (a,b)_r + (b,c)_r + \text{error}. \qquad (2.7)_r$$

The error is of the form

$$\text{error} = O(|a-b| + |b-c|)Q(a,b,c) \qquad (2.8)$$

where Q is a measure of the interaction of waves connecting a to b with those connecting b to c. The precise definition of this notion is as follows:

A wave connecting a to b interacts with those waves connecting b to c which it would intersect. Thus the right wave connecting a to b interacts with the left wave connecting b to c while the left wave connecting a to b does not interact with the right wave connecting b to c. On the other hand the two left waves interact with each other if and only if one of them is a shock wave, and similarly for the right waves.

The numerical measure Q is defined as follows:

$$Q(a,b,c) = \sum |(a,b)_i(b,c)_j|, \qquad (2.9)$$

the summation being over all pairs of interacting waves.

We describe now briefly the difference scheme used in [2] to construct approximate solutions. An approximate solution is exact in each strip $ns \leq t < (n+1)s$; at $t = ns$ the solution is constant in each interval $(m-2)r < x < mr$. The ratio r/s is chosen to be greater than $\mathrm{Max}(\rho, |\lambda|)$; this guarantees that the waves generated at the points of discontinuity $x = (m-2)r$ and $m\,r$ don't interact with each other in the above strip, so that the exact solution there consists of constant states separated by centered rarefaction waves or shocks, see figure 2.5:

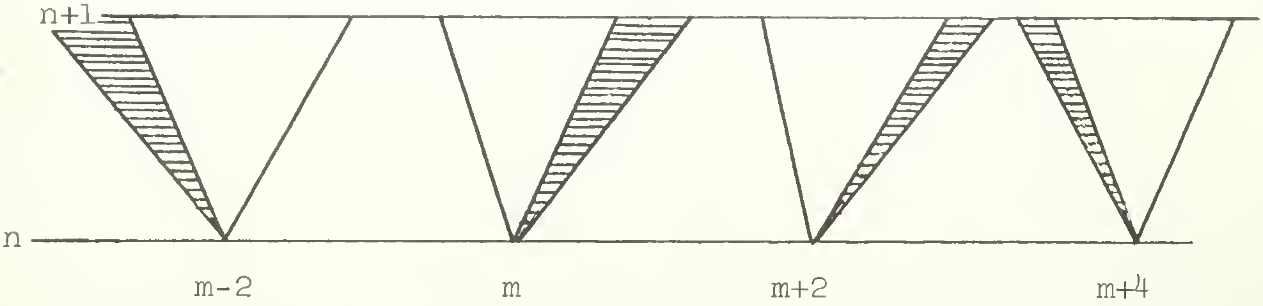

Fig. 2.5

We shall call these waves elementary waves of the approximate solution under discussion.

At $t = (n+1)s$ the exact solution is not piecewise constant; an approximate piecewise constant solution is obtained according to the following recipe:

Pick a random number α_{n+1}, equidistribution[1] in $(-1,1)$, and set the value of u at time $(n+1)s$ in the interval $(m-1)r < x < (m+1)r$ to be equal to the value of the exact solutions constructed in the strip at $t = (n+1)s$ and $x = (m+\alpha_{n+1})r$:

---

[1] Equidistribution means that for every subinterval I of $(-1,1)$

$$\lim \frac{S(N,I)}{N} = \frac{1}{2} \text{ length of } I,$$

where $S(N,I)$ is the number of α_n, $n < N$ which lie in I.

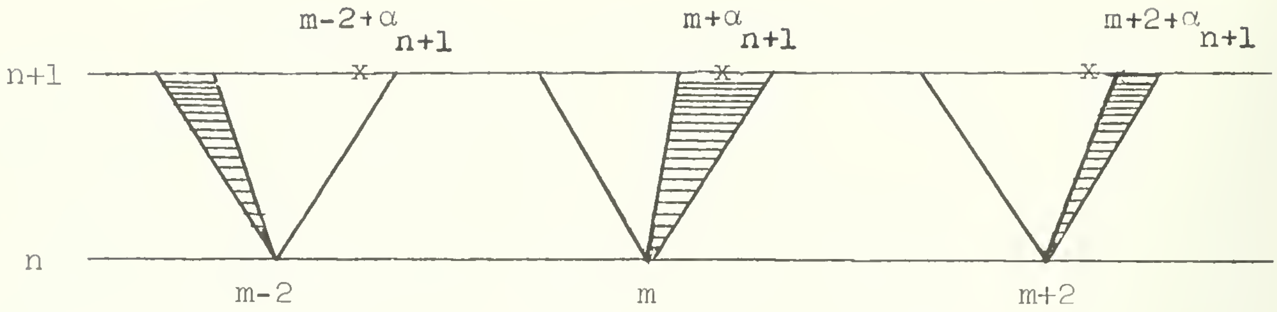

Fig. 2.6

In [2] the following theorem was proved:

Suppose the initial values satisfy the following conditions:

i) The oscillation ε of the initial data is small.

ii) $\eta = \varepsilon$ (total variation of data) is small.

Then there exists a sequence r and s tending to zero, r/s constant subject to the above inequality, such that the approximate solutions tend to a weak solution of the conservation equations (1.1) for almost all random sequences[1] $\{\alpha_n\}$. The oscillation of this solution is $O(\varepsilon)$.

A basic tool in the proof of this theorem is a careful analysis of the interaction of elementary waves. This interaction takes place in the diamond-shaped region $\Delta_{n,m}$ whose vertices S, W, E, N are

$$S = (m+\alpha_{n-1}\ n-1) \quad , \quad N = (m+\alpha_{n+1}, n+1)$$

$$W = (m-1+\alpha_n, n) \quad , \quad E = (m+1+\alpha_n, n).$$

---

[1] In [2] the random sequence depended on m as well as n; it is easy to show that one can dispense with the randomization in m.

The point (m,n) is called the center of $\Delta_{m,n}$. The collection of all diamonds covers the x,t plane, $t > 0$.

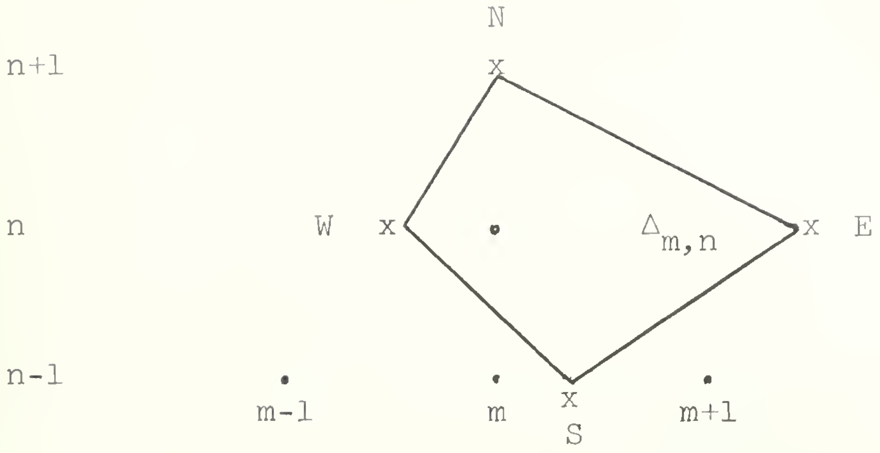

Fig. 2.7

The elementary waves originating at time $n-1$ at the points $m-1$ and $m+1$ enter $\Delta_{m,n}$, interact there, and give rise to two waves emerging from the center (m,n) and propagating to the right and left, respectively.

According to (2.7) the magnitude of each of the waves emerging from Δ equals the sum of the magnitudes of the waves belonging to the same family which enter Δ, plus an error of size (2.8). However when two waves of the same family enter Δ and one of them is a rarefaction wave, the other a shock, as illustrated in figure 2.7, cancellation takes place. The amount C_{right} of right waves cancelled in Δ is defined to be

(2.10) $$C_{\text{right}}(\Delta) = \frac{1}{2}\left(|\alpha|+|\beta|-|\alpha+\beta|\right)$$

where α and β are the magnitudes of the right waves which enter Δ.

We can summarize these facts as a

Conservation law for diamonds:

Denote by $E^{\pm}_{\text{right}}(\Delta)$ the sum of the magnitudes of all rarefaction, respectively shock waves moving to the right which enter Δ, and denote by $L^{\pm}_{\text{right}}(\Delta)$ the magnitude of the right wave which leaves Δ. Then

$$(2.11)_r \quad L^{\pm}_{\text{right}}(\Delta) = E^{\pm}_{\text{right}}(\Delta) \mp C_{\text{right}}(\Delta) + O(\varepsilon)Q(\Delta)$$

where $C_{\text{right}}(\Delta)$ denotes the amount of right wave cancelled in Δ, ε is the oscillation of the solution in Δ and $Q(\Delta)$, defined by (2.9), measures the interaction of all waves entering Δ. A similar law holds for left waves.

Let Λ be the union of a collection of diamonds, $E^{\pm}_{\text{right}}(\Lambda)$ and $L^{\pm}_{\text{right}}(\Lambda)$ the sum of the magnitude of the right waves entering and leaving Λ, respectively. By adding up the conservation laws $(2.11)_r$ for the diamonds constituting Λ we get the following

Conservation Law:

$$(2.12)_r \quad L^{\pm}_{\text{right}}(\Lambda) = E^{\pm}_{\text{right}}(\Lambda) \mp C_{\text{right}}(\Lambda) + O(\varepsilon)Q(\Lambda),$$

where $C_{\text{right}}(\Lambda)$ is the sum over all diamonds of Λ of the amounts of right waves cancelled in each diamond Δ; ε is an upper bound for the oscillation in each diamond, and

(2.13) $$Q(\Lambda) = \sum_{\Delta \text{ in } \Lambda} Q(\Delta).$$

A similar conservation law holds for left waves.

Let I be any interval on the initial line, $\Lambda(I)$ the domain of determinacy of I, see figure 2.8:

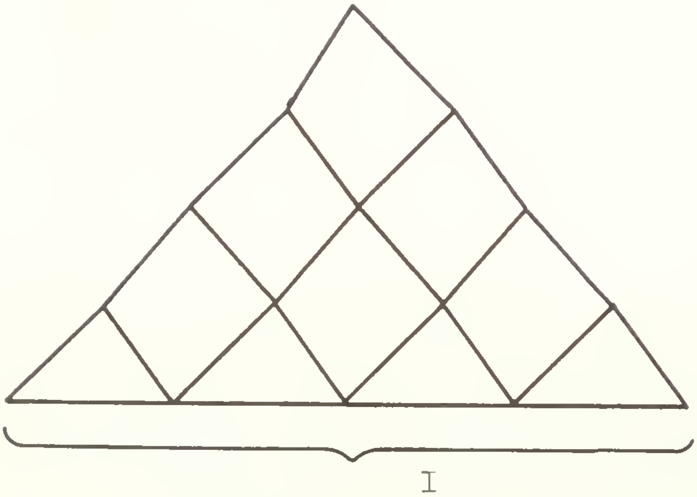

Fig. 2.8

Denote by $V(I)$ the total variation of w and z along I; it has been shown in [2], inequality[1] on top of page 709, that if

$$\varepsilon V(I) \leq \text{const},$$

then

(2.14) $$Q(\Lambda(I)) \leq 2V(I)^2.$$

This estimate is basic for this paper.

Since rarefaction waves have positive magnitude, $L^+_{\substack{r\\ \ell}}$ are nonnegative. Thus we deduce from $(2.12)^+$ that

(2.15) $$C_{\substack{r\\ \ell}}(\Lambda(I)) \leq E^+_{\substack{r\\ \ell}}(\Lambda(I)) + O(\varepsilon)Q(\Lambda(I)).$$

---

[1] Note that the Q in [2] is different from Q in this paper, which is denoted there as D.

The waves entering $\Lambda(I)$ are just the elementary waves emanating from the initial interval I, and therefore we can estimate the sum of their magnitudes in terms of the total variation of w and z along I:

$$(2.16) \qquad E^{+}_{\text{right}}(\Lambda(I)) + E^{+}_{\text{left}}(\Lambda(I)) \leq O(V(I))$$

Using (2.16) and (2.14) to estimate the right side of (2.15) we get

$$(2.17) \qquad C_{\text{right}}(\Lambda(I)) + C_{\text{left}}(\Lambda(I)) \leq O(V(I)) + O(\varepsilon V^2(I)).$$

In addition to the conservation law for domains composed of diamonds we need one for domains bounded partly by characteristics and shocks. For this purpose we define for each approximate solution two families of approximate characteristic curves, propagating to the left and right respectively, as follows:

In each strip $(n-1)s < t < ns$, where the approximate solution is exact, an approximate characteristic is a straight line segment $\chi^{(n-1)}$ which is an exact characteristic or shock and which originates at time $t = (n-1)s$ from the center of some diamond. The continuation of this characteristic is a similar line segment $\chi^{(n)}$ located in the next strip and issuing from the center of that diamond Δ in which the upper endpoint of $\chi^{(n-1)}$ lies:

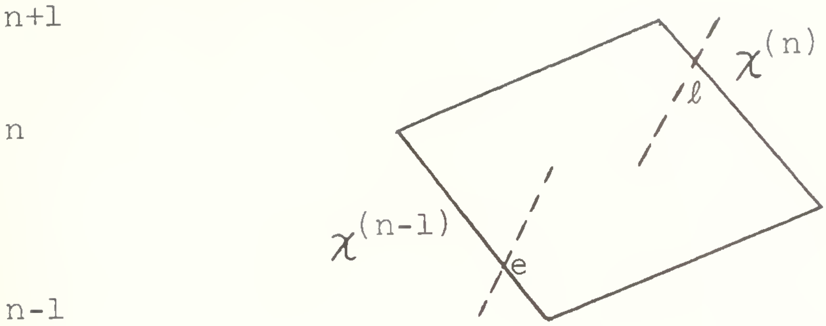

Fig. 2.9

The slope of $\chi^{(n)}$ has to be determined so that, except for cancellations, there is a conservation of waves on both sides of the characteristic. We will make this more precise after giving the rule in each of the six possible situations depending on the nature and relative strength of the two waves of the same family entering Δ; the trivial case when only one wave of the family under discussion enters Δ is subsumed under our classification if we regard the second wave as having strength zero.

In figure 2.10 only right waves are shown, two entering, one leaving.

Since each wave could be a shock or a rarefaction, there are eight possible combinations. According to (2.7) the magnitude of the wave leaving is the sum of the magnitudes of the waves entering, plus an error term; this shows that if we disregard the error term, then if the two entering waves are both shock or both rarefaction, the wave leaving is also a shock or rarefaction. Thus out of the eight possibilities only six are dynamically permissible.

$\chi^{(n-1)}$ and $\chi^{(n)}$ are indicated by dotted lines; for the sake of definiteness we assume that χ^{n-1} enters from the left.

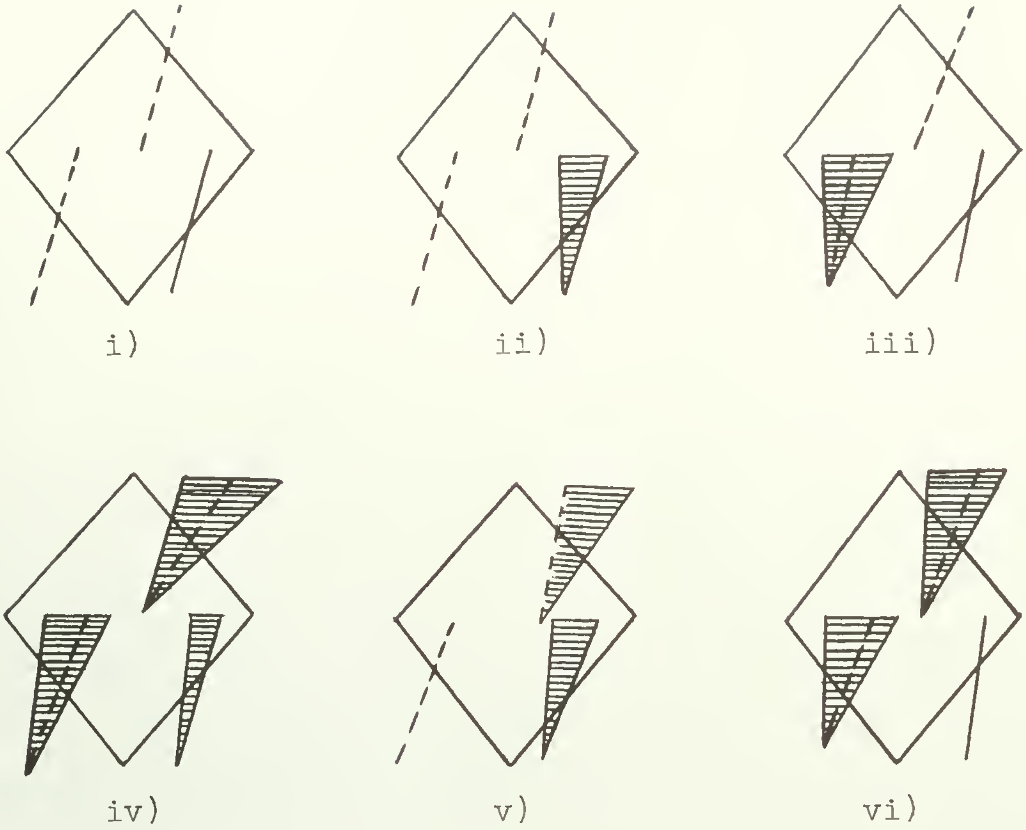

Fig. 2.10

In cases i) - iii) $\chi^{(n)}$ is the shock wave leaving Δ. In cases iv) - vi) the choice of $\chi^{(n)}$ is dictated by the following principle:

Rarefaction waves cannot cross approximate characteristics.

In case iv) this means that the magnitude of the rarefaction wave leaving Δ to the right (left) of $\chi^{(n)}$ equals the sum of the magnitudes of the rarefaction waves entering Δ to the right (left) of $\chi^{(n-1)}$.

In case v) $\chi^{(n)}$ is the left edge of the rarefaction wave leaving Δ.

In case vi) there are two subcases: if the magnitude of the rarefaction wave leaving Δ is greater than the magnitude of the wave which enters Δ to the left of $\chi^{(n-1)}$, we choose $\chi^{(n)}$ so that the magnitude of the wave leaving Δ to the left of $\chi^{(n)}$ equals the magnitude of the wave entering Δ to the left of $\chi^{(n-1)}$. Otherwise we choose $\chi^{(n)}$ to be the right edge of the rarefaction wave leaving Δ.

According to (2.7) the magnitude of the rarefaction wave leaving Δ differs by $O(\varepsilon)Q(\Delta)$ from the sum of the magnitudes of the waves entering Δ; it can happen therefore in case iv) and in the first subcase of vi) that $\chi^{(n)}$ prescribed above lies outside the rarefaction fan leaving Δ; in such a case we redefine $\chi^{(n)}$ to be the right edge of the wave.

Suppose two different approximate characteristics $\chi^{(n-1)}$ and $\psi^{(n-1)}$ enter the same Δ, $\psi^{(n-1)}$ to the right of $\chi^{(n-1)}$. It is not hard to see that our prescription puts $\chi^{(n)}$ to the right of $\chi^{(n)}$; thus approximate characteristics of the same family cannot cross. Of course it can happen - in fact one of the main conclusions of this paper is that it must happen - that $\psi^{(n)}$ coincides with $\chi^{(n)}$. Thus initially distinct approximate characteristics may coalesce.

Connect the endpoint of $\chi^{(n-1)}$ to the beginning of

$\chi^{(n)}$; the resulting piecewise linear curve divides Δ into two halves:

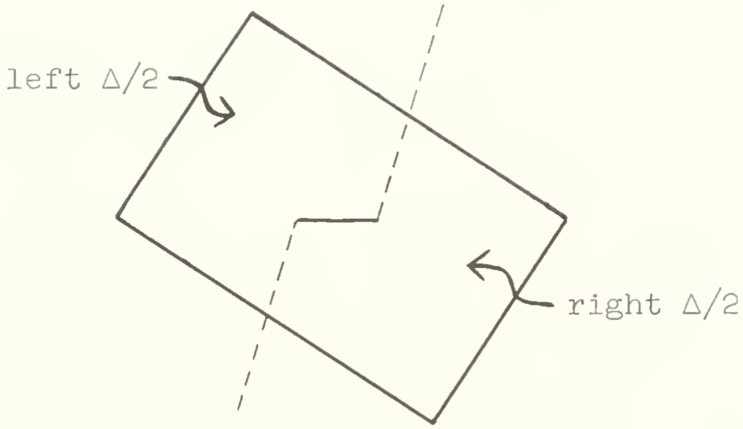

Fig. 2.11

In each half, denoted as $\Delta/2$, the following conservation law for right waves holds:

$$(2.18)^{+} \qquad L^{+}(\Delta/2) = E^{+}(\Delta/2) - C^{+}(\Delta/2) + O(\varepsilon)Q(\Delta/2)$$

and

$$(2.18)^{-} \qquad S(\Delta/2) + L^{-}(\Delta/2) = E^{-}(\Delta/2) + C^{-}(\Delta/2) + O(\varepsilon)Q(\Delta/2)$$

where:

$L^{\pm}(\Delta/2)$ and $E^{\pm}(\Delta/2)$ denote the amount of $\pm$ right wave entering and leaving $\Delta/2$ through that part of its boundary which is part of the boundary of Δ. If $\chi^{(n-1)}$ or $\chi^{(n)}$ is a shock wave, we count it as belonging to neither half of Δ; for this reason we introduce the nonpositive quantity $S(\Delta/2)$ to denote the amount of shock wave which leaves $\Delta/2$ and becomes part of the interior boundary.

$C^{\pm}(\Delta/2)$ are nonnegative quantities and denote the absolut amount of shock and rarefaction waves respectively cancelled in $\Delta/2$. These are no longer equal; nevertheless the following is true:

(2.19) $\quad Q(\text{right } \Delta/2) + Q(\text{left } \Delta/2) = Q(\Delta)$

$(2.19)^+$ $\quad C^+(\text{right } \Delta/2) + C^+(\text{left } \Delta/2) = C(\Delta)$

and

$(2.19)^-$ $\quad C^-(\text{right } \Delta/2) + C^-(\text{left } \Delta/2) \leq C(\Delta)$.

where right $\Delta/2$ and left $\Delta/2$ denote the two halves of Δ. The reason why the sign of inequality may hold in $(2.19)^-$ is that some of the cancellation may involve a shock on the characteristic and that is not counted as part of either right $\Delta/2$ or left $\Delta/2$.

The conservation law may be verified in each half of Δ in all six cases.

If there are $2, 3, \ldots, m$ approximate characteristics crossing Δ, they divide Δ into $m + 1$ parts; in each part a conservation law like (2.18), (2.19) holds. Let Q_j be the amount of interaction present in the j^{th} fraction, $j = 1, \ldots, m + 1$. It can be verified that

$$\sum Q_j = O(Q(\Delta)).$$

Let Λ be the union of a collection of diamonds and diamonds cut by, say, right approximate characteristics.

Adding up the conservation laws (2.11) for the diamonds and $(2.18)^{\pm}$ for fractional diamonds we get the following conservation law:

$$(2.20)^{+} \qquad L^{+}(\Lambda) = E^{+}(\Lambda) - C^{+}(\Lambda) + O(\varepsilon)Q(\Lambda)$$

and

$$(2.20)^{-} \qquad L^{-}(\Lambda) + S(\partial\Lambda) = E^{-}(\Lambda) + C^{-}(\Lambda) + O(\varepsilon)Q(\Lambda)$$

where $S(\partial\Lambda)$ denotes the amount of shock wave which leaves Λ and becomes part of its boundary.

We shall use these conservation laws to derive an estimate for the total variation of w and z along an approximate characteristic χ. We estimate first the total amount of shock entering χ and then the total amount of left wave crossing χ.

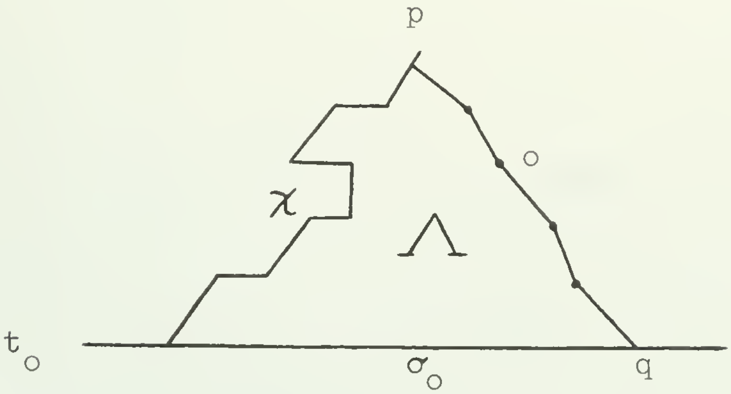

Fig. 2.12

Denote by p the point where the characteristic segment at the upper end of χ intersects the boundary of the diamond from which it is emerging. Construct the curve σ which connects the point p to a point q on the line $t = t_o$ on which χ starts so that σ consists of NE sides of diamonds along which t decreases from p to q, see figure 2.12. Denote by Λ the union of diamonds and half diamonds between χ and σ and lying above $t = t_o$. The only waves which enter Λ enter through the part of the boundary labeled σ_o which lies on $t = t_o$; therefore we conclude from the conservation law (2.20) that

$$(2.21) \qquad |S(\chi)| \leq |E_r^-(\sigma_o)| + O(\varepsilon)Q(\Lambda).$$

Denote by Λ_o the union of whole diamonds which are totally contained in Λ and apply the conservation laws (2.12) for left waves to Λ_o. The left waves which leave Λ_o on the boundary adjacent to χ are the ones which enter the diamonds in which χ is situated. We call their sum the total amount of left wave crossing χ, and denote it by $X^{\pm}(\chi)$. Since all the left waves entering Λ_o do so through σ_o we conclude from the conservation law $(2.12)_\ell$ that

$$(2.22) \qquad X^+(\chi) + |X^-(\chi)| \leq E^+_{left}(\sigma_o) + |E^-_{left}(\sigma_o)| + O(\varepsilon)Q(\Lambda_o).$$

Next we make use of the last two estimates to estimate the total variation of w and z along either side of χ.

It is easily verified by inspection of all cases described in figure 2.10 that the change in w across right waves between the points where χ enters and leaves Δ does not exceed $S(\Delta/2) + C^+(\Delta/2) + O(\varepsilon)Q(\Delta/2)$. A further change in w is due to the left shock waves crossing χ, but the magnitude of this is $O(\varepsilon^2)$ times the strength of that shock wave. Since we have already obtained bounds for the total amount of shock entering χ, the total amount of left wave crossing χ, and the total amount of cancellation in the diamonds containing χ, we obtain an estimate for the total variation of w on either side of χ. A similar estimate holds for the total variation of z along either side of χ.

As an easy corollary we estimate the total variation of the speed of the segments $\chi^{(n)}$ which make up χ. If $\chi^{(n)}$ lies in a rarefaction wave, its speed $\dot{\chi}^{(n)}$ equals $\rho(w,z)$, w and z being the value of the variables at the point where $\chi^{(n)}$ leaves Δ; if $\chi^{(n)}$ is a shock, its speed, according to [4], is approximately the average value of ρ on the two sides; more precisely

$$\dot{\chi}^{(n)} = \frac{\rho(w_-,z_-)+\rho(w_+,z_+)}{2} + O((w_+-w_-)^2 + (z_+-z_-)^2). \tag{2.23}$$

Of course (2.23) is valid in both cases. It follows easily from (2.23) that *the total variation of* $\dot{\chi}^{(n)}$ *along* χ *is* O(*total variation of* w *and* z *along both sides of* χ).

We need one more conservation law, and for this purpose we introduce the concept of an *approximate time-like curve*.

We recall from the theory of hyperbolic equations that a curve $x = x(t)$ is called time-like if its speed of propagation $\dot{x}$ lies between the two characteristic speeds:

$$(2.24) \qquad \lambda < \dot{x} < \rho .$$

We assume inequality (2.24) to hold for all the left and right speeds λ and ρ at any point in the flow field.

Given an approximate solution we construct an approximation to a given time-like curve as follows: τ_{appr} is a straight line segment in each strip $(n-1)s < t < ns$, whose speed equals $\dot{x}(ns)$, starting at the center of that diamond which contains the endpoint of the previous segment.

Let τ be an approximate time-like curve; denote by $\Delta(\tau)$ the union of all diamonds which contain points of τ. It is an immediate consequence of (2.24) that left waves enter $\Delta(\tau)$ only from the right and leave it only on the left; the converse is true for right waves. The amount of right and left waves entering $\Delta(\tau)$ is defined to be the amount of right and left wave crossing τ; these quantities are denoted by $X^{\pm}_{right}(\tau)$ and $X^{\pm}_{left}(\tau)$ respectively, where the superscript $\pm$ indicates rarefaction and shock waves[1].

The conservation law in question concerns domains bounded by an approximate, say, right characteristic χ, an approximate time-like curve τ and an interval σ_0 on a line $t = \text{const}$, as indicated schematically in figure 2.13:

---

[1] It follows from the conservation law (2.12) that the amount of wave which actually crosses τ, i.e. emerge on the other side of $\Delta(\tau)$ is $X(\tau)$ minus the amount of waves cancelled in $\Delta(\tau)$, modulo $O(\varepsilon)Q(\Delta(\tau))$.

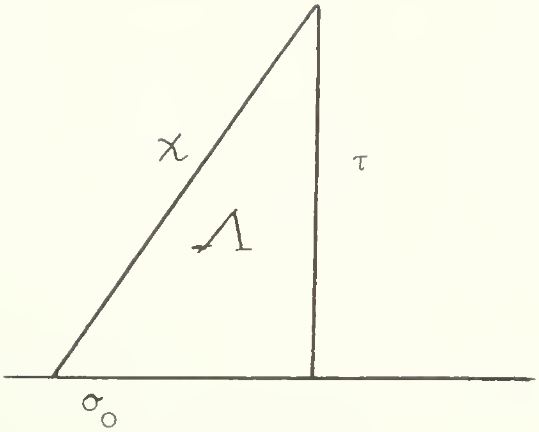

Fig. 2.13

Consider the region Λ consisting of all diamonds which are entirely contained between χ, τ and σ_o. According to the foregoing, left waves enter Λ only through the bottom and right boundaries, and leave it only through the left boundary. Applying the conservation law (2.12) we conclude:

$$(2.25) \qquad |X_{\text{left}}(\sigma_o)| \leq |X_{\text{left}}(\chi)| + C(\Lambda) + O(\varepsilon)Q(\Lambda),$$

where $|X_{\text{left}}|$ is an abbreviation for $X^+_{\text{left}} + |X^-_{\text{left}}|$.

We could define approximate space-like curves σ analogously but that is not necessary; we define instead an approximate space-like curve to be a continuous curve which is a union of WN or WS sides of diamonds. The amount of right and left wave crossing σ is defined in the obvious fashion and is denoted as $X^{\pm}_{\text{left}}(\sigma)$ and $X^{\pm}_{\text{right}}(\sigma)$. We abbreviate by the symbol $|X(\sigma)|$ the total amount of wave crossing σ:

$$|X(\sigma)| = X^+_{\text{left}}(\sigma) + |X^-_{\text{left}}(\sigma)| + X^+_{\text{right}}(\sigma) + |X^-_{\text{right}}(\sigma)|.$$

The total amount of wave crossing a space-like curve σ is almost, but not quite, equal to the total increasing and decreasing variations of w and z along σ; the discrepancy is due to the fact that z changes across a right shock wave, although only by an amount proportional to the cube of the shock strength. From this estimate for the discrepancy we deduce the following relations:

Increasing variation of w along σ =

$$(2.26) \qquad = X^+_{\text{right}}(\sigma) + O(\varepsilon^2)|X(\sigma)|.$$

Decreasing variation of w along σ =

$$= X^-_{\text{right}}(\sigma) + O(\varepsilon^2)|X(\sigma)|.$$

Similar relations hold between the variation of z and the total amount of left wave crossing σ.

We summarize for easy reference the various estimates, see (2.17), (2.21), (2.22), derived with the aid of our conservation laws:

Theorem 2.1: The quantities listed below are bounded by $O(V(\sigma)) + O(\varepsilon)V(\sigma)^2$, where $V(\sigma)$ is the total variation of w and z along a space-like curve σ whose domain of determinacy includes the pointset in question:

(A) $C_{right}(\Lambda) + C_{left}(\Lambda)$, *the amount of wave cancelled in a union* Λ *of diamonds*.

(B) $S(\chi)$, *the amount of shock wave which enters a characteristic*.

(C) *The total variation of* w *and* z *along either side of a characteristic* χ.

(D) *The total variation of the speed* $\dot{\chi}$ *of a characteristic*.

(E) $Q(\Lambda)$, *the total amount of interaction in* Λ.

We derive now some further estimates on the strength of shocks:

The analysis which led to the conservation law (2.18) shows that, mod $O(\varepsilon)Q(\Delta)$, the strength of a shock crossing Δ can increase at most by the amount of shock which enters Δ, and can decrease at most by the amount of rarefaction waves cancelled in Δ.

Let χ be an approximate characteristic and denote by Λ the union of all diamonds through which χ cuts during the time $t_i \leq t \leq t_f$. Denote by $\mathrm{Str}\,\chi(t)$ the *strength*, i.e. the absolute value of the magnitude, of the shock on χ at time t. The estimate given above on the increase or decrease of the strength of χ upon crossing each diamond yields this inequality:

$$(2.27) \qquad \mathrm{Str}\,\chi(t_i) - C(\Lambda) + O(\varepsilon)Q(\Lambda) \leq \mathrm{Str}\,\chi(t_f) \leq$$

$$\leq \mathrm{Str}\,\chi(t_i) + |E^-(\Lambda)| + O(\varepsilon)Q(\Lambda),$$

$E^-(\Lambda)$ being the amount of shock entering .

Using the definition (2.9) of interaction Q we can establish the following lower bound for $Q(\Lambda)$:

(2.28) $$\mathrm{Str}_{\min} |E^-(\Lambda)| \leq Q(\Lambda)$$

where $\mathrm{Str}_{\min}$ is the minimum strength of χ in (t_i, t_f).

We give now an estimate of the amount of shock $E^-(\Lambda)$ which enters χ. Let I be an interval at time $t = t_i$ containing $\chi(t_i)$; suppose that there are approximate right characteristics χ_ℓ and χ_r issuing from the endpoints of I which do not run into χ for $t \leq t_f$. Denote by Λ_I the set of points between χ_ℓ and χ_r, $t_i \leq t \leq t_f$. According to the conservation law $(2.20)^-$

(2.29) $$|L^-(\Lambda_I)| \leq |E^-(\Lambda_I)| + O(\varepsilon)Q(\Lambda_I).$$

Since Λ_I is bounded by approximate right characteristics, shock waves enter it only through I, i.e.

$$E^-(\Lambda_I) = X_r^-(I).$$

On the other hand since χ_ℓ and χ_r do not run into χ, $\chi(t_f)$ is one of the waves leaving Λ_I; so we have from (2.29) that

(2.30) $$\mathrm{Str}\, \chi(t_f) \leq |X_r^-(I)| + O(\varepsilon)Q(\Lambda_I).$$

3. The construction of exact solutions as a limit of approximate solutions.

The basic estimate of [2] is that the approximate solutions u_k described in Section 2 satisfy the inequality

$$(3.1) \qquad \operatorname*{Var}_x u_k(x,t) \leq \text{const Var } u_k(x,0).$$

Furthermore, as was shown in [2], this inequality implies that the approximate solutions $u_k(\cdot,t)$ are uniformly continuous in the L_1^{loc} topology. It follows from this, with the aid of Helly's theorem, that a suitably selected subsequence of $\{u_k\}$ converges in the L_1^{loc} topology with respect to x, to some limit u, uniformly in t. It was shown in [2] that for almost all choice of the sequence of random points $\{\alpha_n\}$ the limit u is a weak solution of the originally given conservation laws.

The limit u is of bounded variation in x for each t; it follows that for each t, $u(x,t)$ is a continuous function of x except for a denumerable set of x. It is not hard to show, using Fubini's theorem, that the set of discontinuities has measure zero in x,t.

The purpose of this section is

a) To construct characteristics and shocks for exact solutions.

b) To show that the slope of such a characteristic equals the characteristic speed at that point, and that the speed of a shock satisfies the Rankine-Hugoniot equations.

c) To prove approximate conservation of variation for regions bounded by space-like, time-like and characteristic arcs.

We shall construct characteristic curves χ for an exact solution u as limits of approximate characteristics χ_k constructed in Section 2 for the approximate solutions u_k of which u is the limit:

Lemma 3.1: Let $x = \chi_k(t)$ be an approximate characteristic, whose initial point $(\chi_k(t_k), t_k)$ tends to (x_o, t_o). Then we can select a subsequence of k such that $\chi_k(t) \rightarrow \chi(t)$ uniformly on every bounded time interval. The limit $\chi(t)$ is Lipschitz continuous.

Proof: By construction, χ_k satisfies

$$(3.2) \qquad |\chi_k(t_1) - \chi_k(t_2)| \leq (r_k/s_k)|t_1 - t_2| + 2r_k \quad .$$

Since in the difference scheme r_k/s_k is independent of k, (3.2) gives a uniform bound on the oscillation of $\chi_k(t)$; the existence of a uniformly convergent subsequence follows then as in the argument for the Arzela-Ascoli theorem.

(3.2) implies that r_k/s_k is a Lipschitz constant for the limit function χ.

We recall from Section 2 that an approximate characteristic consists of straight line segments, and that the point of origin of the n^{th} segment depends on the choice of a random parameter α_n. The following lemma is crucial for showing that for a.a. choices of the set of random parameters, the limit characteristic curve has the correct speed of propagation.

Lemma 3.2: a) Let the exact characteristic χ be the limit of the approximate ones χ_k, and suppose that for all k large enough and all t in $[t_*,t^*]$ the speeds $\dot{\chi}_k$ satisfy

$$(3.3)_k \qquad a \leq \dot{\chi}_k \leq b.$$

Suppose that the sequence $\{\alpha_k\}$ of random numbers is equidistributed; then the speed of the limit characteristic also lies between a and b:

$$(3.3) \qquad a \leq \dot{\chi} \leq b$$

b) Suppose that χ and ψ are two characteristics, limits of the sequences χ_k and ψ_k respectively. Suppose that for all k large enough and all t in $[t_*,t^*]$ the speeds of the approximate characteristics satisfy

$$(3.4) \qquad \dot{\chi}_k \leq \dot{\psi}_k + c \ .$$

Suppose that the sequence $\{\alpha_k\}$ is equidistributed; then

$$(3.5) \qquad \chi(t^*) - \chi(t_*) \leq \psi(t^*) - \psi(t_*) + c(t^*-t_*).$$

Proof: Suppose that for a particular n, α_n satisfies

$$\alpha_n < \frac{as}{r}$$

where a is a lower bound for $\dot{\chi}_k$; then the $(n+1)^{st}$ segment of χ_k starts at distance r to the right of the beginning of the n^{th} segment. Let n_1 and n_2 be integers such that $t_* = n_1 s_k$ and $t^* = n_2 s_k$ denote by $S = S(n_1,n_2,a)$ the number of α_n, $n_1 \leq n \leq n_2$ for which

$$\alpha_n < \frac{as}{r} \tag{3.6}$$

It follows that χ_k contains at least S jumps to ther right, at most $n_2 - n_1 - S$ jumps to the left, and so

$$r_k(2S-(n_2-n_1)) \leq \chi_k(t^*) - \chi_k(t_*). \tag{3.7}$$

Equidistribution means that

$$\frac{S}{n_2-n_1} = [1+o(1)](1 + \frac{as}{r})/2 \tag{3.8}$$

which implies that

$$r_k[2S-(n_2-n_1)] = [a+o(1)]s_k(n_2-n_1).$$

Substituting this into (3.7) we get that

$$a(t_2-t_1) + o(1) \leq \chi_k(t_2) - \chi_k(t_1) .$$

Letting k tend to infinity we obtain that

$$a(t_2-t_1) \leq \chi(t_2) - \chi(t_1),$$

which implies the left side of inequality (3.3); the right side follows similarly, and so does part b) of lemma 3.2.

It follows from the weak law of large numbers that almost all sequences $\{\alpha_n\}$ are equidistributed. In an actual calculation we need not rely on this property of random sequences but would preferably employ a particularly evenly distributed sequence, such as the fractional part of $n\,\theta$, where θ is a quadratic irrationality.

In what follows we assume that the sequence $\{\alpha_n\}$ is equidistributed.

Lemma 3.3: Let $\chi = \lim \chi_k$ be a characteristic curve. Then we can select a subsequence for which

$$\lim \dot{\chi}_k(t) = \dot{\chi}(t) \tag{3.9}$$

for all but a countable set of t.

Proof: According to theorem 2.1, part (D), the total variation of $\dot{\chi}_k$ over a given t interval is uniformly bounded for all k. Also, the functions $\dot{\chi}_k(t)$ are uniformly bounded. It follows from these facts by Helly's theorem that for a suitable subsequence the limits

$$\lim \dot{\chi}_k(t) = s(t)$$

and

$$\lim \int_0^t |d\dot{\chi}_k(\tau)| = \overline{s}(t)$$

exist for every t. We claim that (3.9) holds at every point of continuity t_o of $\overline{s}(t)$. To see this choose, given any positive δ, t_* and t^* close enough to t_o so that

$$t_* < t_o \leq t^*$$

and

$$\int_{t_*}^{t^*} |d\dot{\chi}_k| < \delta \ .$$

Clearly assumption $(3.3)_k$ of lemma 3.2 is fulfilled with $a = s(t_o) - \delta$, $b = s(t_o) + \delta$:

$$s(t_o) - \delta < \dot{\chi}_k < s(t_o + \delta).$$

So we conclude by that lemma that

$$s(t_o) - \delta < \dot{\chi}(t) < s(t_o) + \delta \quad .$$

From these bracketings of $\dot{\chi}_k$ and $\dot{\chi}$ and the arbitrariness of δ relation (3.9) follows.

Having shown that the slope of every characteristic χ is the limit of the slopes of the approximate characteristics χ_k of which χ is the limit, we turn to showing that likewise the value of the solution u on either side of χ is the limit of the values of the approximate solutions u_k on either side of χ_k. The key fact here is the one-sided equicontinuity of the approximate solutions:

Lemma 3.4: We can select a subsequence such that the following statement is true for all but a countable set of values t:

Given any positive ε there is a $\delta = \delta(\varepsilon, t)$ such that for all $0 < y < \delta$

$$|u_k(\chi_k(t) \pm y, t) - u_k(\chi_k(t) \pm 0, t)| < \varepsilon. \tag{3.10}$$

Before presenting the proof of lemma 3.4 we draw some conclusions from it:

Corollary: For all but the aforementioned countable set of t

$$\lim u_k(\chi_k(t) \pm 0, t) = u(\chi(t) \pm 0, t). \tag{3.11}$$

Proof of corollary: According to part (C) of theorem 2.1 the total variation of $u_k(\chi_k(t) \pm 0, t)$ is uniformly bounded. So by Helly's theorem we can select a subsequence for which the limit on the left of (3.11) exists; denote the value of this limit as $U^{\pm}(t)$. We want to show that $U^{\pm}(t) = u(\chi(t) \pm 0, t)$.

By the triangle inequality

$$\begin{aligned} |U^{\pm}(t) - u(\chi(t) \pm y, t)| \leq {} & |U^{\pm}(t) - u_k(\chi_k(t) \pm 0, t)| + \\ & + |u_k(\chi_k(t) \pm 0, t) - u_k(\chi(t) \pm y, t)| + \\ & + |u_k(\chi(t) \pm y, t) - u(\chi(t) \pm y, t)|. \end{aligned} \tag{3.12}$$

The first term on the right is by definition of U less than ε for k large enough; the second term is by (3.10), less ε provided that $y < \delta/2$ and k is so large that $|\chi_k(t) - \chi(t)| < y$. Finally, according to the convergence theorems of [ ] quoted at the outset of this section the integral of the last term with respect to y over $\delta/4 \leq y \leq \delta/2$ tends to zero as $k \to \infty$; this implies that there is a sequence y_k, in $(\delta/4, \delta/2)$ such that

$$|u_k(\chi(t) + y_k, t) - u(\chi(t) + y_k, t)|$$

tends to zero. So for k large enough the above quantity is less than ε; applying this to the last term on the right in (3.12) with $y = y_k$ we obtain

$$|U^{\pm}(t) - u(\chi(t) + y_k, t)| < 3\varepsilon.$$

Since ε is arbitrary and y_k can be taken as close to zero as desirable, by choosing δ small enough, it follows that $U^{\pm}(t) = u(\chi(t) \pm 0, t)$. This completes the proof of (3.11).

Denote as usual $u(\chi(t) + 0, t) - u(\chi(t) - 0, t)$ by $[u]$.

Theorem 3.1: Along each characteristic χ, the Rankine-Hugoniot relations

$$(3.13) \qquad \dot{\chi}(t) = \frac{[f]}{[u]} = \frac{[g]}{[v]}$$

are satisfied at all points where $[u], [v] \neq 0$, and the characteristic relation

$$(3.14) \qquad \dot{\chi}(t) = \rho(u(\chi(t),t))$$

at all points where $[u] = 0$, again with the possible exception of a denumerable set of t.

Proof: By construction, the slopes of the approximate characteristics satisfy the Rankine-Hugoniot, respectively characteristic relations. Letting k tend to ∞ and using the limit relations (3.9) and (3.11) we obtain (3.13) and (3.14) as limits of $(3.13)_k$ and $(3.14)_k$ respectively.

We turn now to the proof of lemma 3.4. We recall from section 2, equation (2.9) and (2.10), the quantities

$$(3.15) \qquad C_{right}(\Delta),\ C_{left}(\Delta),\ Q(\Delta)$$

denoting the amount of right and left wave cancelled in the diamond Δ, and the amount of wave interaction in Δ; these quantities depend of course on the approximate solution u_k. We denote by $dC_r^{(k)}$, $dC_\ell^{(k)}$ and $dQ^{(k)}$ the measures which assign the quantities (3.15) to the center of Δ.

We further recall from section 2, equation (2.22) and (2.18), the quantities

$$(3.15)_\chi \qquad X^{\pm}(\chi\cap\Delta) \text{ and } S(\Delta)$$

denoting the amount of left wave crossing χ in Δ, and the amount of right shock entering χ in Δ. We denote by $dX^{(k)}(\chi)$ and $dS^{(k)}(\chi)$ the measure which assigns the quantities (3.15) to the center of each Δ intersected by χ; the superscript k calls attention to the dependence of these measures on u_k.

We select a subsequence so that $\chi_k \to \chi$ and so that the limits

$$(3.16) \qquad dC_r^{(k)} \to dC_r,\ dC_\ell^{(k)} \to dC_\ell, dQ^{(k)} \to dQ$$

$$(3.17) \qquad dX^{\pm k}(\chi_k) \to dX^{\pm}(\chi),\ dS^{(k)}(\chi_k) \to dS(\chi)$$

exist in the w* topology for measures.

Let t_o be any value of t for which the point $\chi(t_o)$, t_o has measure zero with respect to

$$dC_r + dC_\ell + dQ$$

and such that t_o has measure zero with respect to

$$dX^+(\chi) + |dX^-(\chi)| + dS(\chi).$$

The excluded t_o form at most a countable set.

Let γ be any sufficiently small positive number; we shall denote by $O(\gamma^a)$ any quantity less than $K\gamma^a$, K independent of γ and k. For t_o as above we can find a neighborhood $V(\gamma)$ of $\chi(t_o)$, t_o such that for all k large enough

(3.18) *The amount of wave in* u_k *cancelled in* V *is less than* γ^3

(3.19) *The amount of left wave crossing* χ_k *in* V *is less than* γ^3

(3.20) *The amount of shock entering* χ_k *in* V *is less than* γ^3

(3.21) *The amount of interaction in* V *is less than* γ^3.

We shall apply the conservation laws and derive some further estimates on waves in V. The final upshot will be that the oscillation of u_k at t_o on either side of χ_k along an interval whose length depends only on γ is small if γ is small. This is inequality (3.10) of lemma 3.4.

Consider the domain Λ, shown on figure 3.1, bounded by χ_k, an interval σ located at $t < t_o$ and some time-like line τ, all located in V:

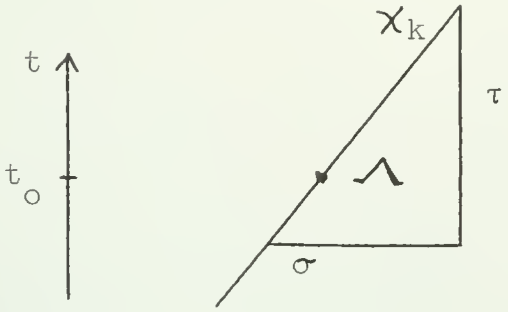

Fig. 3.1

We apply the law of conservation of left waves to Λ. Since (3.18), (3.19) and (3.21) estimate the amount of cancellation, the amount of left wave crossing χ_k, and the amount of interaction, respectively, we obtain, see (2.25), an estimate on the amount of left wave crossing σ :

$$|X_{\text{left}}(\sigma)| \leq 3\gamma^3. \tag{3.22}$$

Using once more the law of conservation of left waves for a domain bounded by any right characteristic ϕ_k contained in a subdomain of V (to be again denoted by V), a spacelike line σ as above and a timelike line, and using (3.18) and (3.21) we deduce as before that

$$|X_{\text{left}}(\phi_k)| \leq O(\gamma^3). \tag{3.23}$$

(3.22) is an estimate for the total amount of left waves crossing σ ; we wish to estimate also the total amount of right waves crossing σ. We start with proving the following

Assertion: Let ϕ_k be any approximate characteristic located in V which contains at time t a shock of strength $> \gamma^{3/2}$. Then the strength of ϕ_k at any later time $t' > t$ is at least $\gamma^{3/2} - \gamma^3$, at most $4\gamma^{3/2}$.

Proof: According to (3.18) the amount of shock cancelled in V is at most γ^3; by (2.27) the strength of ϕ_k at times before it has attained the strength $\gamma^{3/2}$ is at most $\gamma^{3/2} + \gamma^3$, and afterwards at least

$\gamma^{3/2} - \gamma^3$. To estimate the subsequent strength of ϕ_k from above we denote by $\{\beta\}$ the shocks which enter ϕ_k. Since the strength of ϕ_k is at least $\frac{1}{2}\gamma^{3/2}$, the entry of a shock of strength β causes an interaction in the amount at least $\frac{1}{2}\gamma^{3/2}\beta$. Since by (3.21) the total interaction in V is at most γ^3, by (2.28)

$$(3.24) \qquad \sum \beta \leq 2\gamma^{3/2}.$$

According to the discussion in section 2, the paragraph preceeding relation (2.23), the variation of u_k along either side of a characteristic ϕ_k is bounded by the sum of the total amount of shock entering ϕ_k, the amount of right wave cancelled, and the amount of left wave crossing ϕ_k. These quantities are estimated by (3.24), (3.18) and (3.23). According to (2.23) the total variation of u_k on either side of ϕ_k furnishes an estimate on the total variation of the speed $\dot{\phi}_k$ of ϕ_k. Thus

(3.25) The total variation of the speed of any right characteristic ϕ_k in V which starts with a shock of strength $> \gamma^{3/2}$ is at most $O(\gamma^{3/2})$.

To proceed further we have to distinguish two cases, corresponding roughly to the limit characteristic χ being a shock of strength $\geq \gamma$ or not.

A) There is a right shock α_k contained in u_k of strength $\geq \gamma$ for all k large enough, and α_k

tends to $\chi(t_o), t_o$.

B) V can be chosen so small that for infinitely many k all right shocks of u_k in V have strength less than γ.

We choose $R = R(\gamma)$ so small that a square centered at $\chi(t_o), t_o$ with edge $12R$ is contained in V. We shall refer to $12R$ as the inner diameter of V.

We turn now to case A); we denote by ψ_k that portion of the characteristic through α_k which is contained in V. Assuming that all characteristic speeds are < 1 (easily accomplished by rescaling) we conclude that for $t_o \leq t \leq t_o + 6R$, ψ_k lies in V.

By assumption A) for k large enough the initial point of ψ_k lies close to $\chi(t_o), t_o$. We shall eventually show that ψ_k in fact lies on χ_k. As a first step we show that all characteristics point toward ψ_k; more precisely:

Assertion: There exists a positive constant c such that

$$\rho(u_k(r,t)) + c\gamma \leq \dot{\psi}_k(t) \tag{3.26}$$

for all t satisfying $|t-t_o| < R$ and all r such that

$$\psi_k(t) \leq r < \psi_k(t) + c\gamma R \ . \tag{3.27}$$

Similar estimates hold to the left of ψ_k.

Proof of assertion: Since the strength of ψ_k initially is $\geq \gamma$, it follows by our previous reasoning that its strength is always $\geq \gamma - \gamma^3$. It follows from the genuine nonlinearity of the system under study, see section 1, equation (1.4), that the characteristic speeds of the state u^{right} on the right side of ψ_k satisfies

$$\rho(u^{\text{right}}) + c_o\gamma < \dot{\psi}_k \ . \tag{3.28}$$

This shows that (3.26) holds for r adjacent to ψ_k. We shall prove that (3.26) holds with $c = c_o/2$ for all r satisfying (3.27). For suppose on the contrary that (3.28) is violated at $t = t_k$, $|t_k - t_o| < R$, at some point, say, to the right of ψ_k satisfying (3.27). We abbreviate $\psi_k(t_k)$ by p_k, and denote by r_k the first point to the right of p_k where (3.26) is violated at time t_k. We claim:

There is a right characteristic through (r_k, t_k) whose speed s satisfies

$$s = \dot{\psi}_k(t_k) - c\gamma + O(\gamma^3). \tag{3.29}$$

Proof: According to (3.28), (3.26) holds at the point adjacent to ψ_k; therefore (3.26) is satisfied at the constant state on the left of the waves issuing from r_k, while it is violated at the state on the right. Those two states are separated by a left wave and a right wave; according to (3.22) the strength of the left wave is $O(\gamma^3)$, and therefore across it ρ changes by $O(\gamma^3)$. Since ρ decreases across a right shock wave and increases across a right rarefaction wave, it follows that the right wave through r_k is either a shock wave of strength $O(\gamma^3)$ or a rarefaction wave; in either case (3.21) can be satisfied.

Denote by ϕ_k the right characteristic through (r_k,t_k) with initial slope s, restricted to V:

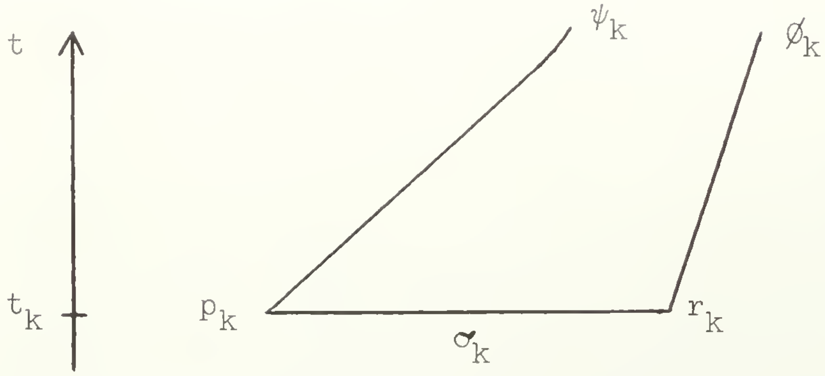

Fig. 3.2

Then (3.29) can be rewritten as

$$\dot{\phi}_k(t_k) = \dot{\psi}_k(t_k) - c\gamma + O(\gamma^3). \tag{3.30}$$

Denote by σ_k the interval $p_k \leq x \leq r_k$, $t = t_k$;

We claim:

(3.31) All right shocks crossing the segment σ_k have strength $< \gamma^{3/2}$ for k large enough.

Proof: Suppose on the contrary that some right shock β_k has strength $\geq \gamma^{3/2}$ for arbitrarily large k. Denote by κ_k the right characteristic issuing through β_k, restricted to V. Since β_k lies between p_k and r_k, its speed satisfies (3.26):

$$\dot{\kappa}_k(t_k) < \dot{\psi}_k(t_k) - c\gamma$$

According to (3.25) the speed of κ_k and of ψ_k varies at most by $O(\gamma^{3/2})$, and so

$$\dot{\kappa}_k(t) < \dot{\psi}_k(t) - c\gamma + O(\gamma^{3/2}) \tag{3.32}$$

for all t.

By contrary assumption $t_k - t_o < R$; R was so chosen that the square with sidelength 12R centered at $\kappa(t_o)$, t_o is contained in V. Since we have assumed that all characteristic speeds are < 1 it follows easily that the portion $t_k < t < t_k + 2R$ of both ψ_k and κ_k are contained in V.

According to inequality (3.32) the speed of κ_k is less than the speed of ψ_k by $c\gamma + O(\gamma^3)$. Therefore according to part b) of lemma 3.2, for k large enough ψ_k and κ_k approach each other at the rate of $c\gamma + O(\gamma^3)$. On the other hand, the distance of ψ_k and κ_k at $t = t_k$ is according to (3.27) less than $c\gamma R$; therefore the distance of ψ_k to κ_k at $t = t_k + 2R$ is less than

$c\gamma R - (c\gamma + O(\gamma^3)2R$ which is negative for γ small enough. Thus ψ_k and κ_k meet at some time between t_k and $t_k + 2R$; since the first shock has strength γ, the second $\gamma^{3/2}$, in the diamond where they meet there is interaction in the amount of $\gamma^{5/2}$, contrary to (3.21).

From the estimate (3.31) we deduce

(3.33) The total variation of $z(x,t)$ along σ_k is $O(\gamma^3)$.

Proof: The Riemann invariant z changes across left waves and right shock waves; according to (3.22) the total amount of the former is bounded by $O(\gamma^3)$. Across a right shock of strength β the change in z is $O(\beta^3)$; therefore the total variation of z across right shocks is bounded by

$$\mathrm{Max}\beta^2 \sum \beta_j .$$

According to (3.31) $\mathrm{Max}\beta$ over σ_k is $\gamma^{3/2}$; as shown in Section 2, $\sum \beta_j$ is bounded à priori by the total variation of the initial data. Thus $\mathrm{Max}\beta^2 \sum \beta_j \leq (\gamma^{3/2})^2$ const; this completes the proof of (3.33).

Next we show that not only the shocks crossing the interval σ_k but all those contained in V and lying between ψ_k and ϕ_k have for $t > t_k$ strength less than $O(\gamma^{3/2})$. To show this we denote by β_n the strength of the strongest right shock wave at the n^{th} time step which lies between ψ_k and ϕ_k. According to the basic theorem governing the interaction of waves, see equations (2.7), (2.8), two shock waves of strength at most β_n can produce a shock wave of strength at most $2\beta_n$ plus an error term. Thus

$$(3.34) \qquad \beta_{n+1} \leq 2\beta_n + Q_n$$

where Q_n is the amount of interaction in the diamond from which β_{n+1} originates.

It is easy to see that in the production of a shock of strength β_{n+1} out of two shocks of strength at most β_n, the amount Q_n of interaction satisfies the inequality

$$(3.35) \qquad (\beta_{n+1}-\beta_n-Q_n)(\beta_n-Q_n) \leq Q_n.$$

Adding $\beta_n - 4Q_n$ to (3.34) we obtain

$$\beta_{n+1} + \beta_n - 4Q_n \leq 3(\beta_n-Q_n);$$

using this to estimate the second factor on the left in (3.35) we get

$$(3.36) \qquad (\beta_{n+1}-\beta_n-Q_n)(\beta_{n+1}+\beta_n-4Q_n) \leq 3Q_n \quad .$$

Denote the maximum of β_n by B; let N be the smallest index for which $\beta_N = B$. We sum (3.36) over $n_k \leq n \leq N$; using the identity

$$(\beta_{n+1}-\beta_n)(\beta_{n+1}+\beta_n) = \beta_{n+1}^2-\beta_n^2,$$

the estimate (3.21)

$$\sum Q_n \leq \gamma^3,$$

the definition of B:

$$\beta_n \leq B, \ \beta_N = B$$

and estimate (3.31):

$$\beta_{n_k} \leq \gamma^{3/2}$$

we obtain that

$$B^2 - \gamma^3 - 6B\gamma^3 \leq O(\gamma^3)$$

which clearly implies that

$$B \leq O(\gamma^{3/2}), \tag{3.37}$$

as asserted.

Next we claim that

$$\dot{\phi}_k(t) \leq \dot{\psi}_k(t) - c\gamma + O(\gamma^{3/2}) \tag{3.38}$$

for all $t \geq t_k$, k large enough.

Proof: According to (3.30), (3.38) is true at $t = t_k$. According to section 2, the changes in the direction of a right characteristic ϕ_k are due to crossings by left waves, cancellations and entering shock waves. The contributions of the first two are according to (3.23) and (3.18) $O(\gamma^3)$; to estimate the effect of shocks entering ϕ_k we distinguish shocks entering from the right and those entering from the left. Those entering ϕ_k from the right decrease the speed of ϕ_k and therefore favor (3.38); so to complete the proof of (3.38) we have to show that the total amount of wave which enters ϕ_k from the left does not exceed $O(\gamma^{3/2})$.

Denote by β_n the strength of the shock wave which enters ϕ_k from the left at the n^{th} time step; since the strength of ϕ_k at that time is at least $\sum_{m<n} \beta_m - C_n$, where C_n is the amount of wave cancelled in ϕ_k up to time n, the quantity

$$\beta_n(\sum_{m<n} \beta_m - C_n)$$

is a lower bound for the interaction at ϕ_k at the n^{th} time step. Summing with respect to n and using the bound (3.21) for the total amount of interaction we get

$$\sum_{m<n} \beta_n \beta_m - \sum C_n \beta_n \leq \gamma^3 .$$

This can be rewritten as

$$\tfrac{1}{2}(\sum \beta_n)^2 - \tfrac{1}{2} \sum \beta_n^2 - \sum C_n \beta_n \leq \gamma^3 \quad .$$

According to(3.18), $C_n \leq \gamma^3$ and according to (3.37) shocks to the left of ϕ_k satisfy $\beta_n \leq O(\gamma^{3/2})$. Using these estimates in the above inequality, and abbreviating $\sum \beta_n$ as $\sum$ we obtain

$$\frac{1}{2}\sum{}^2 - O(\gamma^{3/2})\sum \leq \gamma^3;$$

from this we deduce that

$$\sum \leq O(\gamma^{3/2})$$

as asserted.

Inequality (3.38) relates the speeds of ψ_k and ϕ_k; it follows then from part b) of lemma 3.2 that ψ_k and ϕ_k approach each other at the rate $c\gamma + O(\gamma^{3/2})$. Since according to (3.27) at time t_k they are less than $c\gamma R$ apart, it follows as before that at some time $t = t_{cross}$ between t_k and $t_k + 2R$, ψ_k and ϕ_k run into each other. We apply now the law of conservation of right rarefaction waves to the region Λ bounded by σ_k and by the segments of ψ_k and ϕ_k between t_k and t_{cross}. No right rarefaction wave leaves Λ across ψ_k or ϕ_k; therefore we conclude that the amount of right rarefaction wave crossing σ_k is equal to $C^{(k)}(\Lambda) + Q^{(k)}(\Lambda)$ which by (3.18) and (3.21) is $\leq 2\gamma^3$.

Since the only kind of wave across which the Riemann invariant w increases is a right rarefaction wave, it follows from what we have just shown that the increase of w across σ_k is at most $O(\gamma^3)$. According to (3.22) the change in z across σ_k is $O(\gamma^3)$; since according to (1.4) ρ is an increasing function of w it follows then that the increase of ρ across σ_k is $O(\gamma^3)$. On the other hand combining (3.28) and (3.29) we get

$$\rho(u^{\text{right}}) < \rho(u(r_k,t_k) - c\gamma + O(\gamma^3)$$

which indicates an increase in ρ across σ_k by $c\gamma + O(\gamma^3)$. This is a contradiction, into which we got by denying (3.26), thus proving the truth of (3.26).

By hypothesis A) the beginning of ψ_k lies close to $\chi(t_o)$, t_o and so inequality (3.26) is applicable to the states on either side of χ_k. We conclude from that inequality and lemma 3.2b) that χ_k and ψ_k approach each other at the rate of $c\gamma$ and, since initially they are not far apart they must meet in V for k large enough. When they meet a shock of strength γ would enter χ_k, contradicting assumption (3.20). The only way out of this contradiction is that ψ_k itself lies on χ_k.

Applying (3.26) to χ_k we conclude that all characteristics within a distance of $c\gamma R$ of χ_k, and for $|t-t_o| < R$, approach χ_k at the rate of $c\gamma$. Thus all characteristics ϕ_k which originate at $t = t_o$ inside the interval

$$(\chi(t_o)-c\gamma R, \chi(t_o)+c\gamma R)$$

must for k large enough run into χ_k. Applying the law of conservation of right shock waves to the region bounded by χ_k, ϕ_k and an interval σ on $t = t_o$ we conclude that all but $2\gamma^3$ of the amount of a right shock crossing σ must enter either χ_k or ϕ_k. By (3.20) no more than

γ^3 can enter χ_k; on the other hand the strength of ϕ_k cannot exceed γ^2 since then there would be more than γ^3 amount of interaction with χ_k when ϕ_k runs into χ_k, contrary to (3.21). This shows that the amount of right shock crossing σ is at most $\gamma^2 + O(\gamma^3)$.

Finally we deduce from the conservation of right rarefaction waves that the total amount of right rarefaction wave crossing σ is $O(\gamma^3)$. Since the total variation of w and z along σ can be estimated in terms of the total amount of wave crossing, we deduce that the total variation of w and z on either side of χ_k at $t = t_o$ is less than $\gamma^2 + O(\gamma^3)$; but then so is the total oscillation. Since γ may be taken arbitrarily small, this completes the proof of lemma 3.4 in case A.

We turn now to case B; here by assumption all right shocks are weaker than γ. It follows from this and from estimate (3.21) for the interactions that the amount of shock entering any right characteristic is at most $O(\gamma)$. Using the arguments leading to the proof of estimate (3.25) we conclude:

The total variation of the speed $\dot{\phi}_k$ *of any right characteristic in* V *is less than*

(3.39) $$K\gamma$$

K *independent of* k *and* γ.

We use this to prove that all characteristics point away from each other. More precisely:

Assertion: Choose as before $R = R(\gamma)$ so small that a square of side length $12R$ centered at $\chi(t_o)$, t_o is contained in V. Let (a,t_1) and (b,t_1) denote two points in V, with $t_1 \leq t_o$ and

$$(3.40) \qquad \chi(t_1) \leq a < b < \chi(t_1) + 6K\gamma R$$

let $\phi_k^{(a)}$ and $\phi_k^{(b)}$ denote approximate right characteristics through these points; then

$$(3.41) \qquad \dot{\phi}_k^{(a)}(t_1) \leq \dot{\phi}_k^{(b)}(t_1) + 3K\gamma .$$

A similar inequality holds to the left of χ.

Proof: Suppose (3.41) is violated; then, since by (3.39) characteristic speeds oscillate only $K\gamma$, we find that for all $t \geq t_1$,

$$(3.42) \qquad \dot{\phi}_k^{(a)}(t) > \dot{\phi}_k^{(b)}(t) + K\gamma .$$

According to lemma 3.2, part b), this implies that for k large enough $\phi_k^{(a)}$ and $\phi_k^{(b)}$ approach each other at the rate $K\gamma$. Since at time t_1 they are less than $6K\gamma R$ apart, it follows that they must intersect at time less than $t_1 + 6R$. Assuming that all characteristic speeds are < 1, the intersection lies in V. We apply now the conservation law for right waves to the region

Λ_k bounded by $\phi_k^{(a)}$, $\phi_k^{(b)}$ and the interval (a,b) at $t = t_1$. We conclude that the amount right rarefaction wave crossing (a,b) is bounded by $C^{(k)}(\Lambda_k) + Q^k(\Lambda_k)$ which according to (3.18) and (3.21) is $\leq 2\gamma^3$. The amount of right shock wave crossing (a,b) is bounded by $2\gamma^3$ plus the amount which enters the bounding characteristics $\phi_k^{(a)}$ or $\phi_k^{(b)}$. Since the strength of these shocks cannot exceed γ, and the amount of cancellation on them is at most γ^3, the amount entering them cannot exceed $2\gamma + \gamma^3$, and so the amount of right shock crossing (a,b) is at most $2\gamma + 3\gamma^3$. Combining this with (3.22) according to which the total amount of left wave crossing (a,b) is $O(\gamma^3)$, we conclude that the total variation of u_k in (a,b) is $O(\gamma)$; in particular then the speed of the characteristics at a and b differ by less than $K\gamma$, contradicting (3.42) and proving our assertion.

We turn now to deriving an inequality in the direction opposite to (3.41). For this purpose we look at the characteristics $\phi_k^{(y)}$ issuing from the points

$$\chi_k(t_o-R) + y,\ t_o - R \tag{3.43}$$

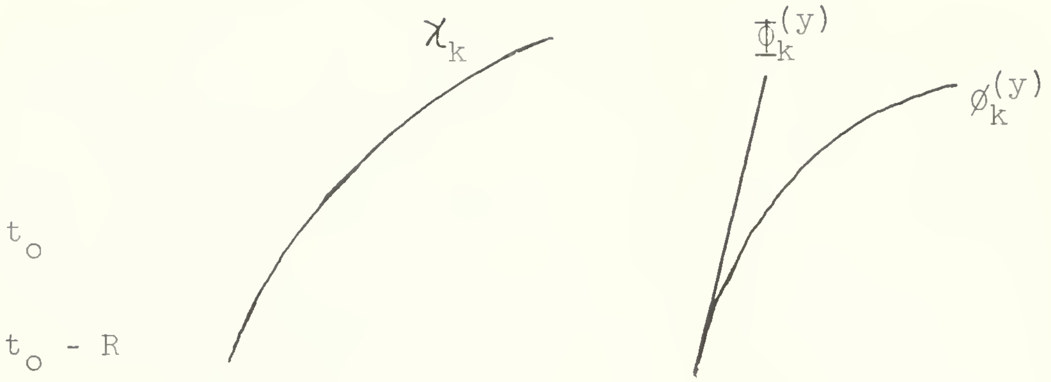

Fig. 3.3

There may be actually a whole fan of characteristics issuing from point (3.43).

We denote by $\Phi_k^{(y)}$ the tangent line to $\phi_k^{(y)}$ at the point (3.43).

Using (3.41) with $a = \chi_k(t_o-R)$, $b = \chi_k(t_o-R) + y$ we conclude that

$$(3.44) \qquad \dot{\chi}_k(t_o-R) \leq \dot{\phi}_k^{(y)}(t_o-R) + 3K\gamma .$$

By (3.39) $\dot{\chi}_k$ oscillates less than $K\gamma$; using lemma 3.2 part b) we conclude from (3.44) that for K large enough

$$\chi_k(t_o) - \chi_k(t_o-R) - K\gamma R \leq \Phi_k^{(y)}(t_o) - \Phi_k^{(y)}(t_o-R) + 3K\gamma R .$$

Since $\Phi_k^{(y)}(t_o-R) = \chi_k(t_o-R) + y$, we conclude that

(3.45) $$\chi_k(t_o) - 4K\gamma R + y \leq \Phi_k^{(y)}(t_o) \quad .$$

On the other hand, since $\phi_k^{(o)}$ is χ_k, we conclude, again using (3.39), that

(3.46) $$\Phi_k^{(o)}(t_o) \leq \chi_k(t_o) + K\gamma R \quad .$$

We claim that there is a value y_k,

(3.47) $$0 \leq y_k < 6K\gamma R$$

such that

(3.48) $$\Phi_k^{(y_k)}(t_o) = \chi_k(t_o) + 2K\gamma R + O(\gamma^3 R).$$

Proof: As y increases $\Phi_k^{(y)}(t_o)$ jumps to the right only across left shocks, whose magnitude according to (3.22) is $O(\gamma^3)$; and by (3.45) and (3.46) the value of $\Phi_k^{(y)}$ at the left and right endpoint of the interval (3.47) is less, respectively greater than the right side of (3.48).

Denote $\phi_k^{(y_k)}$ as ψ_k; we show that ψ is distinct from χ but nearly parallel to it. Applying once more (3.39) we conclude that for k large enough $\psi_k(t_o)$ differs from $\Psi_k(t_o)$ at most by $K\gamma R$; so from (3.48) we conclude that, mod $O(\gamma^3 R)$

(3.49) $$\chi_k(t_o) + K\gamma R < \psi_k(t_o) < \chi_k(t_o) + 3K\gamma R$$

We use the right portion of this inequality to estimate the average speed of ψ_k; using also the fact that ψ_k lies to the right of χ_k we get

$$\psi_k(t_o) - \psi_k(t_o-R) \leq \chi_k(t_o) + 3K\gamma R - \chi_k(t_o-R),$$

so

$$\frac{\psi_k(t_o) - \psi_k(t_o-R)}{R} \leq \frac{\chi_k(t_o) - \chi_k(t_o-R)}{R} + 3K\gamma.$$

Using (3.39) and lemma 3.2 we conclude that for k large enough the average speed of a characteristic differs from its speed at any point at most by $K\gamma$; so we conclude that

(3.50) $$\dot{\psi}_k(t_o) \leq \dot{\chi}_k(t_o) + 5K\gamma$$

Now suppose that a lies in the range

(3.51) $$\chi_k(t_o) < a < \chi_k(t_o) + K\gamma R;$$

according to the first inequality in (3.49), ψ_k lies to the right of $\phi_k^{(a)}$. Applying (3.41) to $\phi_k^{(a)}$ and to $\phi_k^{(b)} = \psi_k$ (with $t_1 = t_0$) we conclude, using (3.50), that

$$\dot{\phi}_k^{(a)}(t_o) \leq \dot{\psi}_k(t_o) + 3K\gamma \leq \dot{\chi}_k(t_o) + 8K\gamma$$

On the other hand, since $\phi_k^{(a)}$ lies to the right of χ_k, we can apply (3.41) to deduce

$$\dot{\ }_k(t_o) - 3K\gamma \le \dot{\phi}_k^{(a)}(t_o).$$

Thus we have upper and lower bounds on $\dot{\phi}_k^{(a)}$ for all a in the range (3.51).

Since the strength of shocks is $< \gamma$, $\dot{\phi}_k$ differs from $\dot{\rho}(u_k)$ at most by $O(\gamma)$ and so we can conclude that the oscillation of $\rho(u_k)$ in the interval (3.51) is $O(\gamma)$.

According to (3.22) the amount of left wave crossing (3.51) is $O(\gamma^3)$; since right shocks are weaker than γ we conclude that the total variation - and a fortiori the total oscillation - of the left Riemann invariant z over (3.51) is $O(\gamma^3)$.

By assumption the system under study is genuinely nonlinear, i.e. $\rho_w \neq 0$. Therefore we can estimate the total oscillation of the other Riemann invariant w over (3.51) in terms of that of z and ρ and find it to be also $O(\gamma)$. Since γ may be taken arbitrarily small, this completes the proof of lemma 3.4 in case B.

We are now ready to pass to the limit. As stated at the beginning of this section it follows from the uniform bounds derived in [2] on the total variation along space-like curves of the approximate solutions u_k that a subsequence of them converges to some limit u in the L_1^{loc} topology in x, uniformly in t. It follows from the uniform bounds obtained in Section 2 on the amount of right and left wave cancelled that for a further subsequence the measures $dC_r^{(k)}$ and $dC_\ell^{(k)}$ have limits dC_r, dC_ℓ' in the w^* topology. Similarly the limit

$dQ^k \rightarrow dQ$ exists.

Let $\chi_k(t)$ be a sequence of, say, right approximate characteristics, whose initial point $\chi_k(t_k), t_k$ tends to some point x_o, t_o. According to lemma 3.1, a subsequence tends to a limit $\chi(t)$, and according to the corollary to lemma 3.4 the limits

$$\lim u_k(\chi_k(t) \pm 0, t),$$

which we denote as $u(\chi(t) \pm 0, t)$, exist for all but a denumerable set of t. According to Theorem 3.1, $\dot{\chi}(t) = \lim \dot{\chi}_k(t)$ exists for all unexceptional values of t, and is equal to $[f]/[u] = [q]/[v]$ at points of discontinuity, to $\rho(u)$ at all points of continuity.

The functions $u(\chi(t) \pm 0, t)$ as defined above are functions of bounded variation; since they were defined in terms of the values of u_k along χ_k,

$$w(\chi(t)-0,t) - w(\chi(t)+0,t) = |[w]| = \lim \text{Strength } \chi_k(t).$$

According to (2.22) the total amount of left wave $\mathfrak{X}(\chi_k)$ crossing χ_k is bounded uniformly for all k. Therefore its limit

$$X^+(\chi) \quad \text{and} \quad X^-(\chi)$$

exists as a measure on χ.

Clearly the limiting form of Theorem 2.1 holds.

In the estimates derived in Section 4 we shall employ a denumerable set of characteristics, each the limit of approximate characteristics of the same subsequence of the u_k, selected by the diagonal process.

Let χ_1 and χ_2 be a pair of right characteristics, limits of χ_{1k} and χ_{2k}. Denote by σ_{tk} approximate space-like curves constructed by connecting adjoining SW edges of diamonds, starting at time $t = ns$ on χ_{2k} and ending on χ_{1k} at time t^*_k. It follows from the argument given in the proof of lemma 3.2 that as k tends to ∞ the curve σ_{tk} tends to the segment σ_t starting at $\chi_2(t), t$, ending at $\chi_1(t^*), t^*$ travelling with speed[1] $- r/s$.

As in Section 2, we let

$$X_{r}^{\pm(k)}(\sigma_{tk})$$

be the amount of right rarefaction, respectively shock, wave in u_k which crosses σ_{tk}. In the case of a right shock wave lying on an endpoint $\chi_{ik} \cap \sigma_{tk}$ of σ_{tk}, we make the convention

---

[1] By a change of independent variables

$$\bar{t} = t, \; \bar{x} = x + at$$

we can achieve that $\lambda_o = - \rho_o$. It follows then that at all points

$$\rho = |\lambda| + O(\varepsilon).$$

We can choose r/s to be $\rho_o + O(\varepsilon)$ and satisfy the required condition

$$r/s \geq \mathrm{Max}(\rho, |\lambda|);$$

with this choice the speed of a space-like curve along which t is monotonically increasing (decreasing) differs by $O(\varepsilon)$ from the speed of any right (left) characteristic. This fact will be exploited in Section 4.

that the wave does not cross σ_{tk}.

Lemma 3.5. After passing to a subsequence the quantities

$$X_{\overline{r}}^{\pm}(\sigma_t) = \lim X_{\overline{r}}^{\pm(k)}(\sigma_{tk})$$

exist for each t and have bounded total variation as functions of t.

Proof: Denote by $\Lambda_{t_1t_2k}$ the strip bounded by χ_{1k}, χ_{2k}, σ_{t_1k} and σ_{t_2k}. Since the sides are approximate right characteristics, right waves enter only through the bottom, rarefaction waves leave only through the top, and shock waves leave through the top and possibly through the sides. We use these facts in conjunction with the conservation laws of Section 2 to conclude that

$$(3.52) \qquad |X_{\overline{r}}^{\pm(k)}(\sigma_{t_2,k}) - X_{\overline{r}}^{\pm(k)}(\sigma_{t_1,k})| \leq C^{(k)}(\Lambda_{t_1t_2k} + O(\varepsilon)Q^k(\Lambda_{t_1t_2k}) + \textstyle\sum_{i=1}^{2} \displaystyle\int_{t_1}^{t_2} dS^{(k)}(\chi_{ik}).$$

We sum this relation over any set of disjoint time intervals (t_1,t_2) and find that the total variation of $X_{\overline{r}}^{\pm(k)}(\sigma_{tk})$ is bounded by

$$C_r^{(k)}(\Lambda_k) + O(\varepsilon)Q^{(k)}(\Lambda_k) + \sum_1^2 \int dS^{(k)}(\chi_{ik})$$

a quantity which according to Theorem 2.1 is uniformly bounded for all k. Applying Helly's selection principle the conclusion of lemma 3.5 follows.

Corollary: Denote by Λ the strip contained between χ_1,

χ_2, σ_{t_1} and σ_{t_2}, where $t_1 < t_2$. Passing to the limit in the conservation law we obtain the inequality

$$|X_r^{\pm}(\sigma_{t_2})| \leq |X_r^{\pm}(\sigma_{t_1})| + O(\varepsilon)Q(\Lambda). \tag{3.53}$$

The family of space-like intervals σ_t constructed above can be replaced by any other monotonic family, such as the horizontal lines σ'_t, obtained as $\lim \sigma'_{tk}$, the union of adjoining WN edges alternating with NE edges. As before, we deduce the existence of the quantities $X_r^{\pm}(\sigma_t)$ and the boundedness of their variation as functions of t. Similarly we deduce the analogues of the conservation laws (3.53).

Next we wish to estimate the variation of the amount of left wave crossing σ_t and so we apply the law of conservation of left waves to $\Lambda_{t_1t_2k}$. This time left waves may enter also through the right and leave through the left boundary. We obtain

$$\begin{aligned} |X_\ell^{\pm(k)}(\sigma_{t_1k}) - X_\ell^{\pm(k)}(\sigma_{t_2k})| &\leq C^{(k)}(\Lambda_{t_1t_2k}) \\ &+ O(\varepsilon)Q^k(\Lambda_{t_1t_2k}) + \sum_1^2 \int_{t_1}^{t_2} |dX_\ell^{\pm}(\chi_i)|. \end{aligned}$$

As before we conclude that the variation of $X_\ell^{\pm}(\sigma_{tk})$ as function of t is uniformly bounded for all k, and so by Helly's selection principle

$$X_\ell^{\pm}(\sigma_t) = \lim X_\ell^{\pm}(\sigma_{tk})$$

exists for all t for a suitable subsequence, and is of bounded variation as function of t.

We need an estimate not only for the variation but also for the magnitude of $X_\ell^{\pm}(o_t)$; for this purpose we consider the triangular region Γ_{tk} formed by σ_{tk}, χ_{2k} and τ_{tk}, where τ_{tk} is a time-like line starting at $\chi_1(t_k^*)$, t_k^* and ending at some point of χ_2:

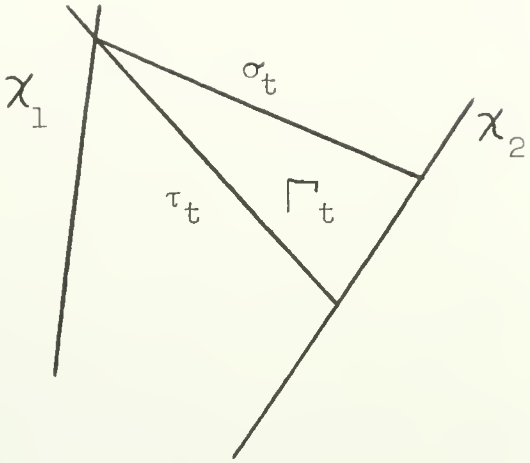

Fig. 3.4

Left waves enter Γ_{kt} only through χ_{2k}; therefore we conclude from the conservation law that

$$|X_\ell^{\pm(k)}(\sigma_{tk})| \leq \int_{t-t_k}^{t} |dX_\ell^{\pm(k)}(\chi_{2k})| + O(\varepsilon)Q^{(k)}(\Gamma_{kt})$$

where $t - t_k$ is the time when τ_{tk} intersects χ_2. Letting k tend to ∞ we obtain

$$(3.54) \qquad |X_\ell^\pm(\sigma_t)| \le \int_{t-t_0}^{t} |dX_\ell^\pm(\chi_2)| + O(\varepsilon)Q(\Gamma_t).$$

We recall from Section 2 that the total amount of wave crossing a curve σ is denoted by $\overset{(k)}{X}(\sigma)$:

$$\overset{(k)}{X}(\sigma) = X_r^{+(k)}(\sigma) + |X_r^{-(k)}(\sigma)| + X_\ell^{+(k)}(\sigma) + |X_\ell^{-(k)}(\sigma)|.$$

According to (2.26), the change of the Riemann invariants w from one end of σ to the other, denoted as $[w](\sigma)$, is

$$[w^{(k)}](\sigma) = X_r^{+(k)}(\sigma) + X_r^{-(k)}(\sigma) + O(\varepsilon^2)X^{(k)}(\sigma),$$

and similarly for z. Actually for z we only need the estimate

$$|[z^{(k)}](\sigma)| \le X_\ell^{+(k)}(\sigma) + |X_\ell^{-(k)}(\sigma)| + O(\varepsilon)X^k(\sigma).$$

We compute now the change in $\rho(u^k)$ across σ; by the mean value theorem

$$[\rho] = \rho_w[w] + \rho_z[z].$$

The oscillation of w and z is $O(\varepsilon)$; so denoting by ρ_w the value of that function at $0,0$ we deduce from the last three relations that

$$(3.55) \qquad [\rho^{(k)}](\sigma) = \rho_w\left\{X_r^{+(k)}(\sigma) + X_r^{-(k)}(\sigma)\right\} + O(\varepsilon)X^{(k)}(\sigma) + O(1)\left\{X_\ell^{+k}(\sigma) + |X_\ell^{-(k)}(\sigma)|\right\}$$

By definition, an approximate characteristic travels either with characteristic speed or shock speed. In either case we have, using the well known relation that shock speed is the average of the characteristic speeds on either side, see [4],

$$\dot{\chi}_k(t) = \rho(u_k(\chi_k(t)+0,t)) - \frac{1}{2}\rho_w \mathrm{Str}\chi_k(t)(1+O(\varepsilon)$$

We recall that the strength (Str) of the shock on χ was defined as the absolute value of the change in w across it.

We can now compute, using (3.55), the difference in the speeds of χ_{1k} and χ_{2k} at the two ends of σ_{tk}:

$$\dot{\chi}_{2k}(t) - \dot{\chi}_{1k}(t_k^*) = \rho_w\{X_r^{+(k)}(\sigma) + X_r^{-(k)}(\sigma) - \frac{1}{2}(\mathrm{Str}\,\chi_1+\mathrm{Str}\,\chi_2)$$

$$+ O(\varepsilon)\{X^{(k)}(\sigma) + \mathrm{Str}\,\chi_1 + \mathrm{Str}\,\chi_2\}$$

$$+ O(1)\{X_\ell^{+(k)}(\sigma) + (X_\ell^{-(k)}(\sigma)|\} .$$

We let k tend to ∞; all quantities tend to a limit for all but a denumerable set of t and we get

Lemma 3.6: For all but a denumerable set of t

(3.56) $$\dot{\chi}_2(t) - \dot{\chi}_1(t^*) = \rho_w\Omega(t) + O(1)\{X_\ell^+(\sigma(t))+|X_\ell^-(\sigma(t))|\}$$

where

(3.57) $$\Omega(t) = X_r^+(\sigma(t)) + X_r^-(\sigma(t)) - \frac{1}{2}(\mathrm{Str}\,\chi_1(t) + \mathrm{Str}\,\chi_2(t))$$

$$+ O(\varepsilon)\{X(\sigma(t)) + \mathrm{Str}\,\chi_1(t) + \mathrm{Str}\,\chi_2(t) .$$

We turn now to estimating the strength of shocks on characteristics. We say that a characteristic χ which is the limit of a sequence of characteristics χ_k runs into another characteristic $\emptyset$ which is the limit of the sequence $\emptyset_k$ for $t \leq t_f$

if all but a finite number of χ_k run into ϕ_k before time t_f.

Let I be an interval at some initial time t_i, containing $\chi(t_i)$; let χ_1, χ_2 be a pair of characteristics issuing from the endpoints of I which do not run into χ for $t \leq t_f$. Then inequality (2.30) holds for infinitely many k; passing to the limit we obtain

$$\text{(3.58)} \qquad \operatorname{Str} \chi(t_f) \leq |X_r^-(I)| + O(\varepsilon)Q(\Lambda_I).$$

The key concepts so far, such as characteristic curve and amount of wave crossing, were obtained as limits of the approximate concepts and therefore may very well depend on the particular subsequence used. It may happen that two characteristic curves χ and ϕ occupy the same position yet they do not run into each other; there may even be a finite amount of wave contained in the infinitely thin strip between them!

We show now that nevertheless the quantities in question serve as upper bounds. According to the discussion at the beginning of Section 2, the Riemann invariant w increases with increasing x across a right rarefaction wave, decreases or remains constant across all others. Therefore for any spacelike curve σ_k

Increasing variation of w_k across σ_k $= X_r^{+k}(\sigma_k)$.

Since increasing variation is lower semi-continuous, we conclude after passing to the limit that

(3.59) Increasing variation of w along $\sigma \leq X_r^+(\sigma)$.

Let $\chi_1, \chi_2, \dots \chi_N$ be a sequence of right characteristics and denote by $\sigma_j(t)$ the horizontal interval between $\chi_{j-1}^{(t)}$ and $\chi_j^{(t)}$; denote by $L(t)$ the interval between χ_1 and χ_N. It follows from the definition that for all but a countable set of t

$$X_r^+(L(t)) = \sum X_r^+(\sigma_j(t)).$$

We shall refer to this property as the additivity of X_r^+.

We turn now to the conservation laws, i.e. inequalities (3.53), (3.54). These were derived as the limits of approximate conservation laws after omitting the amount of wave cancelled or entering the boundary. These inequalities can be improved by replacing $Q(\Lambda)$ by $Q(\Lambda')$ defined as

$$Q(\Lambda') = \lim Q^{(k)}(\Lambda_k).$$

We shall refer to these improved inequalities as (3.53)' and (3.54)'; the important property of $Q(\Lambda')$ is this: Denote by $\Lambda_j(t)$ the strip contained between χ_{j-1} and χ_j and under $\sigma_j(t)$, by $\Pi(t)$ the strip between χ_1 and χ_N and under $\sigma(t)$. Clearly for each k

$$\sum_j Q^{(k)}(\Lambda_{jk}) = Q^{(k)}(\Pi_k),$$

so letting $k \to \infty$ we obtain

$$\sum Q(\Lambda_j') = Q(\Pi') \leq Q(\Pi).$$

We shall call this the additivity of Q.

4. The spreading and cancellation of rarefaction waves.

In Section 1 we proved that if χ_1 and χ_2 are two, say, right characteristics of a differentiable solution, then χ_1 and χ_2 diverge at the rate cW, where W is the increasing variation of the Riemann invariant w between them and c is a constant. This estimate was used to prove that no differentiable space periodic solutions can exist for all time.

In this section we derive a similar estimate for discontinuous solutions, stated as Theorem 4.1. The quantity W is the amount of rarefaction wave between the two characteristics. However, unlike the case of smooth solutions, W can decrease, due to cancellation of rarefaction waves and shocks, so that in this case the estimate on the rate of divergence of the characteristics leads not to a nonexistence theorem but to an estimate on the rate at which W decays in time.

Just as in Section 1, we shall measure the distance between right characteristics at times displaced in the direction in which left waves propagate. This has the effect that the distance between the two right characteristics remains unaffected (approximately) as they pass through left waves.

Let Λ be a strip bounded on the left and right by a pair of right characteristics χ_1 and χ_2. Through the point $\chi_2(t)$, t we construct a spacelike line segment σ_t meeting χ_1 at a time t^*. We give σ_t the speed $\lambda_1 = \lambda(0,0) - O(\varepsilon)$, $O(\varepsilon)$ large enough so that σ_t is spacelike; the choice of $O(\varepsilon)$ determines t^*. Let

$$D(t) = \chi_2(t) - \chi_1(t)$$

and

$$D^*(t) = \chi_2(t) - \chi_1(t^*).$$

Since

$$D^*(t) = -\lambda_o(t^*-t)(1+O(\varepsilon))$$

and

$$D(t) = (\rho_o-\lambda_o)(t^*-t)(1+O(\varepsilon)),$$

the ratio

$$D^*(t)/D(t) = \frac{\lambda_o}{\lambda_o-\rho_o}(1+O(\varepsilon)) \tag{4.1}$$

is approximately constant. Since χ_1 and χ_2 differ by $O(\varepsilon)$,

$$\begin{aligned} D(t^*) &= D(t) + O(\varepsilon)(t^*-t) \\ &= D(t) + O(\varepsilon)D(t); \end{aligned}$$

this implies that

$$D(t^*)/D(t) = 1 + O(\varepsilon). \tag{4.2}$$

Following the proof of (1.12) in Section 1 we obtain

$$\frac{d}{dt} D^*(t) = (\dot{\chi}_2(t^*)-\dot{\chi}_1(t))\frac{\lambda_o}{\lambda_o-\rho_o}(1+O(\varepsilon)). \tag{4.3}$$

According to Lemma 3.6, equation (3.56)

$$(4.4) \qquad \dot{\chi}_2(t^*) - \dot{\chi}_1(t) = \rho_W\Omega(t) + O(1)(X_\ell^+(\sigma_t)+|X_\ell^-(\sigma_t)|).$$

where $\Omega(t)$ is defined by (3.57). In order to estimate the last term in (4.4) we define

$$t_o = O(\varepsilon)D^*(t),$$

with the $O(\varepsilon)$ large enough so that the line segment τ_t from $\chi_1(t^*)$, t^* to $\chi_2(t - \frac{1}{2} t_o)$, $t - \frac{1}{2} t_o$ is timelike.

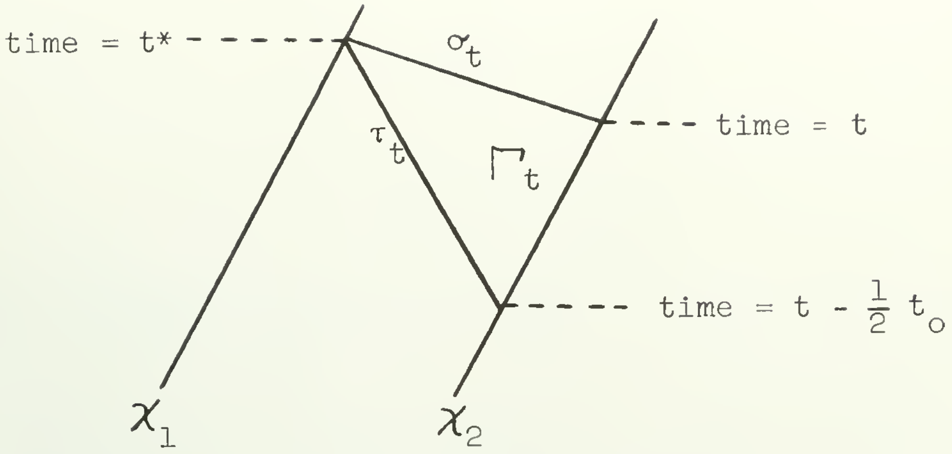

Fig. 4.1

Let Γ_t be the region bounded by σ_t, τ_t and χ_2. Left waves enter Γ_t through χ_2 only and leave through σ_t and τ_t. Thus using (3.54)'

$$(4.5) \qquad |X_\ell^{\pm}(\sigma_t)| \le |L_\ell^{\pm}(\Gamma_t)| \le |E_\ell^{\pm}(\Gamma_o)| + O(\varepsilon)Q(\Gamma_t')$$

$$\le \int_{t-t_o}^{t+o} |dX_\ell^{\pm}(\chi_2)| + O(\varepsilon)Q(\Gamma_t')$$

We define

$$(4.6) \qquad \zeta(t) = \int_0^t dX_\ell^+ + \int_0^t |dX_\ell^-| + O(\varepsilon)Q(\Lambda_t')$$

where $\Lambda_t = \bigcup_{s \leq t} \Gamma_s'$. We can write (4.5) as

$$|X_\ell^\pm(\sigma_t)| \leq \int_{t-t_0}^{t+0} d\zeta(s).$$

We substitute the above estimate and (4.4) in (4.3) and obtain

$$(4.7) \qquad \frac{d}{dt} D^*(t) = \rho_W \Omega(t) \frac{\lambda_0}{\lambda_0 - \rho_0} + O(1) \int_{t-t_0}^{t+0} d\zeta(t).$$

This equation makes sense only when $t - t_0 \geq 0$, which is when $t \geq O(\varepsilon)D^*(t)$. Since

$$(4.8) \qquad D^*(t) = D^*(0) + O(\varepsilon)t,$$

we have $t - t_0 \geq 0$ except for

$$t \leq t_* = O(\varepsilon)D^*(0).$$

Using (4.8) again,

$$D^*(t_*)/D^*(0) = 1 + O(\varepsilon).$$

Since D^* is Lipschitz continuous, it is the integral of its derivative and so integrating (4.7) from t_* to T we obtain

$$(4.9) \qquad D^*(T) = (1+O(\varepsilon))D^*(0) + c\int_{t_*}^T \Omega(t) + O(1)\int_{t_*}^T \int_{t-t_0}^{t+0} d\zeta(s)dt,$$

where

$$c = \rho_w \frac{\lambda_o}{\lambda_o - \rho_o} (1+O(\varepsilon).$$

We wish to eliminate t_* from the single integral in (4.9). To accomplish this we note:

$$|\int_o^{t_*} \Omega dt| \le O(\varepsilon) D^*(0) \sup_t |\Omega(t)|;$$

According to the conservation laws (3.53), (3.54) $|\Omega|$ is bounded by the total variation V_* of the Cauchy data over an interval of length $O(t_*)$; so

$$|\int_o^{t_*} \Omega dt| \le O(\varepsilon V_*) D^*(0).$$

Substituting this in (4.9) gives us the equation

$$(4.10) \qquad D^*(T) = (1+O(\varepsilon)(1+V_*))D^*(0) + c\int_o \Omega(t)dt + O(1)\int_{t_*}^{T}\int_{t-t_o}^{t+o} d\zeta(s)dt.$$

The double integral can be estimated as in Section 1 by a change in the order of integration. With the $t_1(s)$ introduced in Section 1, we have

$$\int_{t_*}^{T}\int_{t-t_o}^{t+o} d\zeta(s)dt \le \int_o^T\int_s^{s+t_1(s)} dtd\zeta(s).$$

Using the estimate $t_1(s) = O(\varepsilon)D^*(s)$ from Section 1, we bound the integral above by

$$O(\varepsilon)\int_o^T D^*(s)d\zeta(s),$$

and (4.10) becomes

$$G(T) - \int_0^T D^*(t)dm \leq D^*(T) \leq F(T) + \int_0^T D^*(t)dm,$$

where $m(t) = O(\varepsilon)\zeta(t)$ and

$$F(T) = (1+O(\varepsilon)(1+V_*))D^*(0) + c\int_0^T \Omega(t)dt,$$

$$G(T) = (1-O(\varepsilon)(1+V_*))D^*(0) + c\int_0^T \Omega(t)dt.$$

Using (1.28) we have

$$(4.11) \qquad G(T) + (1-e^{m(T)})F(T) \leq D^*(T).$$

Substituting the definition of F, G and m into (4.11) gives a lower bound for D^* in terms of V_*, ζ, ε, $D^*(0)$ and Ω. If we assume that

$$(4.12) \qquad \varepsilon V_* + \varepsilon\zeta(T) \text{ is sufficiently small}$$

we can simplify that inequality to

$$(4.13) \qquad c[1\text{-small no.}]\int_0^T \Omega(t)dt \leq D^*(T).$$

Next we proceed to find a lower bound for $\Omega(t)$; using the abbreviations

$$X_r^+(\alpha(t)) = R(t), \ X_r^-(\alpha(t)) = Sh(t)$$

the definition (3.57) of $\Omega(t)$ implies

$$(4.14) \qquad \Omega(t) \geq [1-O(\varepsilon)]R(t) + [1+O(\varepsilon)]Sh(t)$$

$$- \frac{1}{2}[1+O(\varepsilon)][\mathrm{Str}\,\chi_1(t)+\mathrm{Str}\,\chi_2(t)]$$

where Str χ denotes the strength of the shock on χ.

We apply now the law of conservation of right shocks (3.53) to the set $\Lambda(t)$ bounded by σ_t σ_0, χ_1 and χ_2. Dropping the amount of wave cancelled we get

$$(4.15)^- \qquad Sh(t) \geq Sh(0) - O(\varepsilon)Q(\Lambda').$$

Similarly, the conservation law $(3.53)^+$ for rarefaction waves applied to the set bounded by σ_t, σ_T, χ_1 and χ_2 yields

$$(4.15)^+ \qquad R(t) \geq R(T) - O(\varepsilon)Q(\Lambda').$$

Here Λ denotes $\Lambda(T)$.

Finally denote by Str_{max} the maximum strength of χ_1 and χ_2. Substituting $(4.15)^{\pm}$ into (4.14) and the resulting inequality into (4.13) yields, after division by T,

$$(4.16) \qquad c[1\text{-small no.}]R(T) \leq \frac{D^*(T)}{T} + |Sh(0)|$$
$$+ O(\varepsilon)Q(\Lambda) + Str_{max}.$$

We claim now that inequality (4.16) holds without the term Str_{max} on the right.

Proof: We draw a finite number of right characteristics $\chi_1 = \xi_1, < \ldots < \xi_N = \chi_2$ all originating in the interval $[\chi_1(0), {}_2(0)]$, so that the amount of rarefaction wave entering each strip Λ_j bounded by ξ_j, ξ_{j+1} is less than $\varepsilon Q(\Lambda')$. Then, according to the law of conservation of rarefaction waves the amount leaving Λ_j is bounded by $O(\varepsilon)Q(\Lambda')$.

Some of these characteristics ξ_j may run into χ_1 or χ_2 still in Λ; suppose that ξ_{J-1} does but ξ_J doesn't run into χ_1, and that ξ_{K+1} does but ξ_K doesn't run into χ_2. According to the additivity of the amount of rarefaction wave which crosses $\sigma(T)$ between χ_1 and χ_2, $R(T)$ is the sum of the amount crossing between χ_1 and ξ_J, ξ_J and ξ_K, and ξ_K and χ_2. Since χ_1 and ξ_{J-1} have coalesced on $\sigma(T)$, the first amount is equal to what is leaving Λ_j and this is $O(\varepsilon)Q(\Lambda')$; the same estimate holds for the third amount. Thus $R^{JK}(T)$, the amount crossing $\sigma(T)$ between ξ_J and ξ_K differs by $O(\varepsilon)Q(\Lambda')$ from $R(T)$:

$$(4.17) \qquad R(T) \leq R^{JK}(T) + O(\varepsilon)Q(\Lambda').$$

We apply now (3.58) to ξ_J and to ξ_K, and the strip Λ; we obtain that the strength of the shock on each is bounded by $|Sh(0)| + O(\varepsilon)Q(\Lambda)$:

$$(4.18) \qquad Str^J_{max} \leq |Sh(0)| + O(\varepsilon)Q(\Lambda').$$

We apply now inequality (4.16) to the strip bounded by ξ_J and ξ_K and obtain, using (4.18), that

$$c[1\text{-small no.}]R^{JK}(T) \leq D^*_J(T)/T + |Sh(0)| + O(\varepsilon)Q(\Lambda').$$

Using now (4.17) and the fact that $D^*_{JK} \leq D^*$ we obtain

$$(4.19) \qquad c[1\text{-small no.}]R(T) \leq \frac{D^*(T)}{T} + Sh(0) + O(\varepsilon)Q(\Lambda').$$

In the above derivation we have assumed that (4.12) holds.

If all or all but one of the ξ_j run into ξ_1 or ξ_2, then $R(T) \leq O(\varepsilon)Q(\Lambda')$ follows from $(3.53)^+$.

We are now ready to prove the main result of this section:

Theorem 4.1: Let χ_1 and χ_2 be a pair of right characteristics, $\Pi(T)$ the strip bounded by χ_1 and χ_2, $0 \leq t \leq T$. Denote by $L(t)$ the interval $[\chi_1(t), \chi_2(t)]$, its length by $|L(t)|$. Denote by $V_o(\Pi)$ the total variation of the Cauchy data over the domain of determinacy of $\Pi(T)$. Suppose that

(4.20) $\varepsilon V_o(\Pi)$ is sufficently small;

then

(4.21) $$aX_r^+(L(T)) \leq \frac{|L(T)|}{T} + O(\varepsilon)Q(\Pi).$$

The constant a can be taken arbitrarily close to $[1+O(\varepsilon)]\rho_w(0)$ by making the bound in (4.20) small enough.

Note: Clearly this inequality asserts that either characteristics curves spread apart, or the amount of rarefaction wave between them decays.

Proof: We divide Π into alternate strips of predominantly shock, respectively rarefaction flow. More precisely, we divide the initial interval $L(0)$ into a finite number of subintervals $L_j(0)$ so that $|L_j(0)| < \varepsilon$ and $L_j(0)$ contains less than δ_j amount of right shock, respectively rarefaction wave, depending on whether j is even or odd. If the initial data are of bounded variation, we can make the subdivision so fine that

$$\sum \delta_j = \delta$$

is less than any preassigned number.

Denote by ξ_j the characteristic issuing from the j^{th} subdivision point, and by Λ_j the strip bounded by ξ_j, ξ_{j+1}, $L_j(0)$, and $\sigma_j(T_j)$, T_j so chosen that $T_j^* = T$. For j odd,

$$X_r^+(L_j(0)) \leq \delta_j;$$

then according to the law of conservation of rarefaction waves (3.53)'

$$X_r^+(L_j(T)) \leq \delta_j + O(\varepsilon)Q(\Lambda_j'). \tag{4.22$_j$}$$

For j even we apply inequality (4.19); using the fact that $|Sh(0)| < \delta_j$ and that according to the limiting form of Theorem 2.1, $\zeta(T)$ is bounded by $V_o(\Lambda)$ we get from (4.20) that (4.12) is satisfied, so

$$c(1\text{-small no.})R_j(T_j) \leq \frac{D_j^*(T_j)}{T_j} + \delta_j + O(\varepsilon)Q_j'. \tag{4.23$_j$}$$

We apply now the conservation law for rarefaction waves to the region Γ_j shown in figure 4.2:

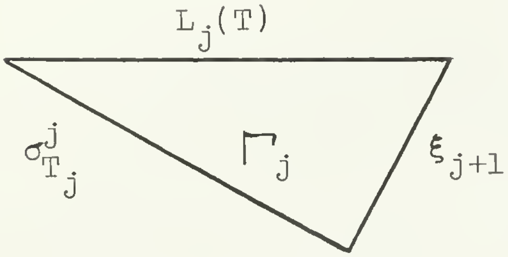

Fig. 4.2

We get

$$X_r^+(L_j(T)) \leq R_j(T_j) + O(\varepsilon)Q(\Gamma'_j).$$

According to (4.1)

$$D_j^*(T_j) = |L_j(T)| \frac{\lambda_o}{\lambda_o-\rho_o} (1+O(\varepsilon)).$$

It is easy to see that since $|L_j(0)| < \varepsilon$,

$$T_j = T(1+O(\varepsilon)).$$

Substituting these three relations into $(4.23)_j$ and recalling that

$$c = \rho_W \frac{\lambda_o}{\lambda_o-\rho_o} (1+O(\varepsilon))$$

we get

$$(4.24)_j \qquad \rho_W(1\text{-small no.})X_r^+(L_j(T)) \leq \frac{|L_j(T)|}{T} + \delta_j + O(\varepsilon)[Q(\Lambda_j)+Q(\Gamma_j)]$$

We add up the inequalities $(4.22)_j$ and $(4.24)_j$; by additivity of amount of wave crossing

$$\sum X_r^+(L_g(T))_1 = X_r^+(L(T))$$

by additivity of length

$$\sum |L_j(T)| = |L(T)|;$$

by construction

$$\sum \delta_j = \delta$$

Finally, according to the additivity of Q

$$\sum Q(\Lambda_j') + Q(\Gamma_j') \leq Q(\Pi).$$

Substituting these relations into $\sum (4.24)_j$ we obtain that

$$\rho_w(1\text{-small no.})X_r^+(L(T)) \leq \frac{L(T)}{T} + \delta + O(\varepsilon)Q(\Pi).$$

Since δ is arbitrary, inequality (4.21) of Theorem 4.1 follows.

We claim that inequality (4.21) holds for any interval $L(T)$, not necessarily bounded by two characteristics; for this purpose we construct two characteristics χ_1 and χ_2 so that $L(T)$ lies between $\chi_1(T)$ and $\chi_2(T)$. Since the speed of propagation of any right characteristic is $\rho(0) + O(\varepsilon)$, we can choose χ_1 and χ_2 so that $\chi_1(T)$ and $\chi_2(T)$ lie within a distance of $O(\varepsilon)T$ of $L(T)$. Denote by $\Pi(L)$ the strip contained between χ_1 and χ_2 so chosen. Next we draw a finite number of characteristics $\chi_1 = \xi_1 < \xi_2 < \ldots < \xi_N = \chi_2$ all originating in the interval $[\chi_1 10), \chi_2(0)]$ so that the amount of rarefaction wave entering each strip bounded by ξ_j, ξ_{j+1} is less than $\varepsilon Q(\Pi)$. Denote by J and K the indices such that $\xi_{J-1}(T)$ does not lie in $L(T)$ but $\xi_j(T)$ does, and similarly so that $\xi_K(T)$ lies in $L(T)$ but $\xi_{K+1}(T)$ doesn't. Thus, since $L(T)$ is contained in $\xi_{J-1}(T)$, $\xi_{K+1}(T)$, we have by additivity of waves crossing that at $t = T$

$$(4.25) \qquad X_r^+(L) \leq X_r^+(\xi_{J-1}, \xi_{K+1}) = X_r^+(\xi_{J-1}, \xi_J) + X_r^+(\xi_J, \xi_K) + X_r^+(\xi_K, \xi_{K+1}).$$

Applying the conservation law (3.53) we get using the bound imposed on $X_r^+(\xi_j,\xi_{j+1})$ at $t = 0$ that at $t = T$

$$(4.26) \qquad X_r^+(\xi_{J-1},\xi_J) + X_r^+(\xi_K,\xi_{K+1}) \leq O(\varepsilon)Q(\Pi).$$

We apply now Theorem 4.1 to the strip bounded by ξ_J, ξ_K; we get an estimate for $X_r^+(\xi_J,\xi_K)$ in terms of $|\xi_J-\xi_K|$ which is $\leq$ L. Combining this inequality with (4.25) and (4.26) we obtain inequality (4.21) for an arbitrary interval. In light of the observation (3.59) we can express this result as follows:

Theorem 4.2: *Let* L *be any x-interval at time* T; *let* Π *be any characteristic strip containing* L, *and suppose that*

$$(4.27) \qquad \varepsilon V_o(\Pi) \text{ is sufficiently small.}$$

Then

$$(4.28) \qquad \rho_w(1\text{-small no.}) \text{ Increasing variation of } w \text{ over } L \leq \frac{|L|}{T} + O(\varepsilon)Q(\Pi).$$

For purposes of easy reference put (4.28) into the following form:

$$(4.29) \qquad \text{Increasing variation of } w \text{ over } L \leq K_o\frac{|L|}{T} + K_1\varepsilon Q(\Pi).$$

The total variation of w is at most twice its increasing variation plus its oscillation; since the total oscillation of U is ε, that of w is $O(\varepsilon)$. We get a similar estimate for the total variation of z. Combining the total variation of $U = \{u,v\}$:

$$(4.30) \qquad \text{Total Variation of } U^{(t)} \text{ over } L \leq K_2\frac{|L|}{T} + K_2\varepsilon + K_2\varepsilon Q(\Pi)$$

5. The existence and decay of solutions with arbitrary Cauchy data near a constant state.

In this section we solve the initial value problem with initial data that are merely measurable, and which differ little from a constant in the L_∞ norm. We show that even if the initial data have infinite total variation, the corresponding solution $U(x,t) = \{u(x,t), v(x,t)\}$ has for every positive t finite variation which decays in time.

For single equations this decay of total variation has been proved by Oleinik, [6]; the decay is a quantitative measure of the smoothing which the initial data undergo, see [4].

Throughout this section C, K and H, with various subscripts, denote constants which depend only on the equation under consideration, and not the solution in question.

Theorem 5.1: We are given bounded, measurable initial data $U(0)$; denote its maximum by η^2:

$$\|U(0)\|_\infty = \eta^2. \tag{5.1}$$

If η is small enough, there exists a solution $U = \{u,v\}$ of the system of conservation equations (1.1) with initial values $U(0)$; this solution satisfies the decay law

$$\text{Total Variation of } U(x,t) \text{ over } L \leq \frac{H_1 |L|}{t} \tag{5.2}$$

for every x-interval L whose length $|L|$ is $\geq t$. Furthermore

$$|U(x,t)| \leq H_2 \eta. \tag{5.3}$$

We shall rely on the following existence theorem and estimates derived in [2]:

Existence Theorem: Suppose that initial data $U(0)$ are prescribed subject to the following restrictions:

$$(5.4)\quad \begin{aligned} &\text{i)}\quad \|U(0)\|_\infty \leq C_1 \\ &\text{ii)}\quad \|U(0)\|_\infty \{\text{Total Variation of } U(0)\}^2 \leq C_2 . \end{aligned}$$

Then there exists a solution $U(x,t)$ with initial value $U(0)$ which satisfies

$$(5.5)\quad \begin{aligned} &\text{i)}\quad \text{Total Variation of } U(t) \leq C_3 \text{ Total Variation of } U(0) \\ &\text{ii)}\quad |U(x,t)| \leq C_4 \|U(0)\|_\infty , \end{aligned}$$

provided that C_1, C_2 are sufficiently small and C_3, C_4 sufficiently large.

It is convenient to assume for what follows and throughout this section that all characteristic speeds are < 1. This permits us to set $r_k = s_k$ in the construction of approximate solutions.

Local Existence Theorem: Suppose the initial data $U(0)$ satisfy condition $(5.4)_i$, but $(5.4)_{ii}$ only over sub-intervals of the x-axis of length δ. Then a solution $U(x,t)$ with initial values $U(0)$ exists for $t < \delta/2$; $(5.5)_i$ is satisfied over intervals of length $|L| < \delta - 2t$.

Sketch of proof: Let J be a subinterval of the x axis of length δ; define $U_J(0)$ to be equal $U(0)$ on J, constant outside of J. These initial data satisfy (5.4); therefore a solution $U_J(x,t)$ exists. Suppose that in the approximation procedure used to construct these solutions we employ the same sequence of time and space increments and the same random numbers α_n. Since the propagation of influence in the approximate solutions is $\frac{r}{s} = 1$, it follows that if J_1 and J_2 are overlapping intervals, the corresponding approximate solutions are equal in the domain of determinacy of the overlap. The same is true for the limits[1] if we choose the same subsequence; this is possible if we employ only a denumerable set of intervals J. We set now

$$U(x,t) = U_J(x,t) \quad \text{in the domain of determinacy of}$$

J. This completes the proof of the local existence theorem.

We break up the proof of Theorem 5.1 into four steps.

Step 1: It suffices to prove the theorem for solutions whose initial data are uniformly smooth.

Reason: Arbitrary bounded initial data $U(0)$ can be approximated by a sequence of mollified data U_n, defined by

$$U_n(0) = U(0) * j_n;$$

here $j_n(x) = nj(nx)$, j some smooth nonnegative function with compact support and total integral 1. Clearly

---

[1] This argument is necessary since we sadly lack a local uniqueness theorem.

i) $\|U_n(0)\|_\infty \leq \|U(0)\|$

ii) $U_n(x,0) \longrightarrow U(x,0)$ a.e.

iii) There is a number M_n such that $|\frac{d}{dx} U_n(x,0)| \leq M_n$.

Suppose we have shown that the solutions U_n satisfy inequalities (5.2) and (5.3); then by Helly's theorem we can select a subsequence which converges to a limit U; this limit has initial values $U(x,0)$ and satisfies inequalities (5.2) and (5.3).

Step 2: It is enough to prove (5.2) for intervals L of length equal to t, with $H_1/2$ in place of H_1.

Reason: If (5.2) holds for intervals of length t, it also holds for intervals of length mt, m an integer. It follows then that (5.2) holds for intervals of length mt, $m > 1$ with H_1 in place of $H_1/2$.

Step 3: Local existence of the solution.

Suppose we are given uniformly smooth data $U(0)$; we choose δ so that the total variation of $U(0)$ over intervals of length δ is ≤ 1. We restrict η to be so small that

$$\eta^2 < c_1, \ 9\eta^2 \leq c_2.$$

Then the hypotheses (5.4) of the local existence theorem are satisfied over interval of length 3δ. The local solution exists in the strip $t \leq \frac{3\delta}{2}$, and satisfies inequalities (5.5) in each right triangle whose hypotenuse lies on the x axis and is 3δ long. Clearly property $(5.5)_{ii}$ implies property (5.3) provided that η is so small that

$$C_4\eta < H_2$$

$(5.5)_i$ asserts that the total variation of $U(t)$ along an interval of length $(3\delta-2t)$ is $\leq C_3$; this implies (5.2) with $|L| = t$ for $t \leq \delta$, provided that

$$(5.6) \qquad C_3 \leq H_1/2.$$

This represents a limitation on H_1.

Step 4: Inductive proof that a solution $U(x,t)$ exists for $t \leq 2^n\delta$ and satisfies (5.2) and (5.3) there.

Step 3 shows that the assertion is true for $n = 1$. Suppose now that (5.2) and (5.3) hold for $t \leq 2^n\delta$. Then the data $U(x,2^n\delta)$ satisfy conditions (5.4) over intervals of length $2^{n+1}\delta$, provided that η is so small that

$$H_2\eta \leq C_1 \ , \quad H_2\eta(2H_1)^2 \leq C_2.$$

If so, according to the local existence theorem the solution $U(x,t)$ exists in the strip $2^n\delta \leq t \leq 2^{n+1}\delta$. What we have to show is that (5.2) and (5.3) are satisfied there.

We note first that it follows from the local existence theorem that $(5.5)_{ii}$ holds; combining this with (5.3) at $t = 2^n\delta$ we conclude that

$$(5.7) \qquad \|U\|_\infty \leq C_4H_2\eta \quad \text{for} \quad t \leq 2^{n+1}\delta.$$

We turn now to estimating the total variation of U for $t \leq 2^{n+1}\delta$. According to inequality (4.30), applied to an interval L of length t, with initial time $t/2$ and $T = t/2$,

$$(5.8) \qquad \text{Total Variation } U(t) \text{ over } L \leq$$

$$\leq K_2 \frac{|L|}{t/2} + K_2\varepsilon + K_2\varepsilon Q$$

$$= 2K_2 + K_2(1+Q)\varepsilon,$$

where Q is the Q-measure of Π, the domain of dependence of L in the strip $(t/2,t)$:

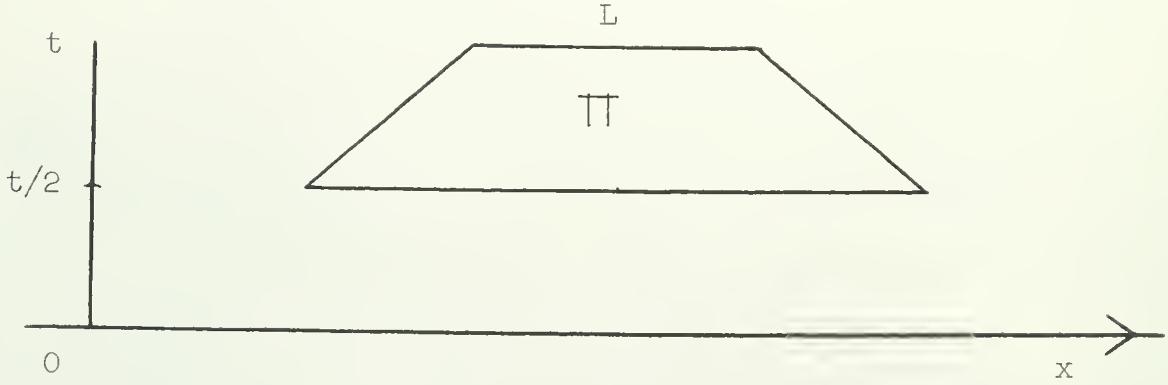

Fig. 5.1

According to the estimate (2.14)

$$(5.9) \qquad Q \leq 2V^2,$$

V the total variation of U over the base of Π. Since the base of Π has length $2t$ and is located at time $t/2 \leq 2^n\delta$, according to induction hypothesis, (5.2) is applicable and gives

(5.10) $$V \leq 4H_1.$$

Combining this with (5.9) we get

(5.11) $$Q \leq 32H_1^2;$$

substituting this into (5.8) as well as the estimate (5.7) for ε we get

(5.12) $$\text{Total Variation } U(t) \text{ over } L \leq 2K_2 + + K_2(1+32H_1^2)C_4H_2\eta.$$

For η small enough the second term on the right in (5.12) is $\leq K_2$; for such η

$$\text{Total Variation } U(t) \text{ over } L \leq 3K_2.$$

Clearly with $H_1 = 6K_2$ the right side is $1/2H_1$ which according to step 2 implies (5.2) for all L of length $\geq t$.

We note that for the validity of estimate (4.30) it was necessary that $V\|U\|_\infty$ be small enough. Using estimates (5.7) and (5.10) we see that the above quantity is $\leq 4H_1C_4H_2\eta$, which is small enough if η is. This completes the proof of (5.2) for $t \leq 2^{n+1}\delta$.

We turn now to proving (5.3) for $t \leq 2^{n+1}\delta$; this calls for estimating $\|U(t)\|_\infty$, or, what is the same, estimating $\|w(t)\|_\infty$ and $\|z(t)\|_\infty$. Inequality (4.29) gives an estimate for the increasing variation of w:

Increasing variation of w on L

$$\leq K_o \frac{|L|}{T} + K_1 \varepsilon Q(\Pi). \tag{5.13}$$

where Π is the domain of dependence of L in $(t-T,t)$. (5.13) alone is not sufficient to estimate $\|w\|_\infty$, unless combined with some absolute estimate for w. Such an absolute estimate is provided by

Lemma 5.1: Let L be any interval located at time t_o; then

$$\int_L |w(x,t)|dx \leq K_4(|L|+t_o)\|U(0)\|_\infty, \tag{5.14}$$

K_4 independent of U; a similar estimate holds for z.

We postpone the proof of Lemma 5.1 and show first how to use it to estimate w. Let x_o, t_o be any point and denote $w(x_o,t_o)$ by w_o; suppose, say, that w_o is positive. Let L be an interval at time t_o whose right endpoint is x_o, t_o. Using (5.13) we get the following lower bound for w on L:

$$w_o - K_o \frac{|L|}{T} - K_1 \varepsilon Q(\Pi) \leq w$$

Integrating this inequality over L and using (5.14) we get, after dividing by $|L|$, that

$$w_o - K_o \frac{|L|}{T} - K_1 \varepsilon Q(\Pi) \leq K_4\left(1 + \frac{t_o}{|L|}\right)\|U(0)\|_\infty.$$

We recall now from (5.1) that $\|U(0)\|_\infty = \eta^2$; choosing $|L| = \eta t_o$ we get from the above inequality that

$$(5.15) \qquad w_o \leq K_4\eta^2 + K_4\eta + K_o \frac{t_o}{T}\eta + \varepsilon K_1 Q(\Pi).$$

We recall that ε is the maximum of $|U|$ in Π and therefore is certainly $\geq$ constant multiple of w_o. Therefore (5.15) cannot furnish an upper bound for w_o unless the coefficient of ε on the right in (5.15) is small enough, i.e. unless $Q(\Pi)$ is small enough. Our only available estimate for Q is (5.9); and our only available estimate for V is (5.2); therefore our best estimate for Q is $2H_1^2$ which certainly isn't small. We have to search for special domains with small Q-measure; we proceed as follows:

Let y, s be any point in the strip $2^n\delta < s < 2^{n+1}\delta$; denote by Γ the rectangle:

$$y - s < x < y + s, \quad 2^{n-1}\delta < t < 2^n\delta$$

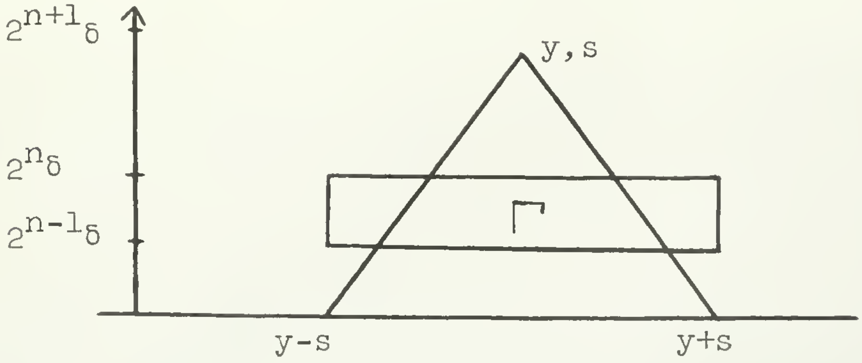

Fig. 5.2

The length of Γ in the x direction is $\leq 2 . 2^{n+1}\delta$; so according to (5.2) the total variation V of U along the lower edge of Γ is $\leq 8H_1$ and hence by (5.9)

$$Q(\Gamma) \leq 128H_1^2. \tag{5.16}$$

We divide Γ by horizontal lines into N congruent rectangles; each subrectangle has vertical width $2^{n-1}\delta/N$; at least one of them, call it Γ_N, must have[1] Q-measure $\leq Q(\Gamma)/N$; by (5.16)

$$Q(\Gamma_N) \leq 128H_1^2/N. \tag{5.17}$$

Denote by t_o the time on the upper edge of Γ_N and let x_o, t_o be any point on that edge which lies in the domain of dependence of y, s. Let L be an interval at time t_o whose right or left endpoint is x_o and whose length is ηt_o; clearly, for $\eta < 1/4$ the domain of dependence Π of L in the time interval $(t_o - T, t_o)$, $T = 2^{n-1}\delta/N$, belongs to Γ_N. So by (5.17)

$$Q(\Pi) \leq Q(\Gamma_N) \leq 128H_1^2/N.$$

Denote as before $w(x_o, t_o)$ by w_o; according to (5.15) with $T = 2^{n-1}\delta/N$ and the above estimate for $Q(\Pi)$

$$w_o \leq K_4\eta^2 + K_4\eta + 2K_oN\eta + \varepsilon K_1 \, 128H_1^2/N. \tag{5.18}$$

A similar estimate holds for $z_o = z(x_o, t_o)$ at any point x_o, t_o in the domain of dependence of y, s. Since u and v are

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[1] We have to avoid having points with positive Q-mass on the horizontal dividing lines.

Lipschitz continuous functions of w and z we get a similar estimate for $U_o = U(x_o,t_o)$:

$$|U_o| \leq KN\eta + \frac{\varepsilon K H_1^2}{N} ,$$

where K is some constant concocted of the K_j, and where we have used that $\eta \leq 1$, $N \geq 1$. Since $t_o < 2^n\delta$, by induction hypotheses (5.3) holds and can be used as estimate for ε. So

$$|U_o| \leq KN\eta + \frac{KH_1^2 H_2\eta}{N} .$$

Clearly if we choose

$$N = 2KH_1^2 C_4$$

and set

$$H_2 = 2KN$$

then the above inequality gives

(5.19) $$|U_o| \leq \frac{H_2\eta}{C_4} .$$

Denote by I_N the domain of dependence of y, s at $t = t_o$. By induction hypothesis (5.2) and (5.3) hold there; it follows that for η small enough the hypotheses (5.4) of the local existence theorem hold, and therefore so does the conclusion (5.5) in the domain of determinacy of I_N. Since y, s belongs to that domain of determinacy we conclude from $(5.5)_{ii}$ and (5.19) that

$$|U(y,s)| \leq C_4\|U(t_o)\|_\infty \leq C_4 \frac{H_2\eta}{C_4} = H_2\eta.$$

Since y, s is an arbitrary point in the strip $2^n\delta < s < 2^{n+1}\delta$, this completes the proof of (5.3).

We turn now to the proof of Lemma 5.1; we need

Lemma 5.2: For any solution U the maximum of w and the minimum of z is reached at $t = 0$.

Proof: Let U_ℓ and U_r denote the states on the left and right respectively in a Riemann initial value problem, and let U_m denote the intermediate state, which is connected to U_ℓ across a left wave, and to which U_r is connected across a right wave. The Riemann invariant w doesn't change across a left rarefaction wave and, according to assumption (2.4) in Section 2, decreases across a left shock. Thus

$$w(U_m) \leq w(U_\ell).$$

Similarly z doesn't change across a right rarefaction wave and according to (2.4) decreases across a right shock. Thus

$$z(U_m) \geq z(U_r).$$

Clearly the same inequalities hold for any state U_m' in the rarefaction waves, if any. This proves Lemma 5.2 for the Riemann initial value problem. Since our approximate solutions were pieced together from solutions of Riemann problems, Lemma 5.2 holds for all approximate solutions, and so for their limits as well.

Remark: If the sign in any of the inequalities (2.4) is changed, the corresponding phase in Lemma 5.2 has to be changed from maximum to minimum, or conversely.

In what follows we shall employ the abbreviation

$$\|U(0)\|_\infty = M. \tag{5.20}$$

Since w and z are smooth functions of U, for any x

$$w(x,0) \leq K|U(x,0)| \leq KM$$

and

$$z(x,0) \geq - K|U(x,0)| \geq - KM.$$

Using Lemma 5.2 we deduce this

Corollary: For any x and any $t > 0$

$$w \leq KM, \quad z \geq - KM. \tag{5.21}$$

This corollary can be stated geometrically: the points (w,z) lie in the second quadrant of the w, z plane with respect to the point $(KM,-KM)$. Since the mapping

$$w,z \rightarrow u, v \tag{5.22}$$

is smooth, it maps this quadrant into some angle with vertex at the image of $(KM,-KM)$. Replacing if necessary the variables u, v by suitable linear combinations we can achieve that the angle to which u, v is confined is the first quadrant. The analytical form of this statement is:

$$- KM \leq u, \quad - KM \leq v; \tag{5.23}$$

here we have used the fact that (5.22) maps the origin into the origin, and we have enlarged the constant K.

It follows immediately from (5.23) that

(5.24) $$|u| < u + 2KM, \quad |v| \leq v + 2KM.$$

We take now the original conservation equations (1.1):

$$u_t + f_x = 0, \quad v_t + g_x = 0,$$

add them and integrate them over the trapezoid shown on figure 5.3:

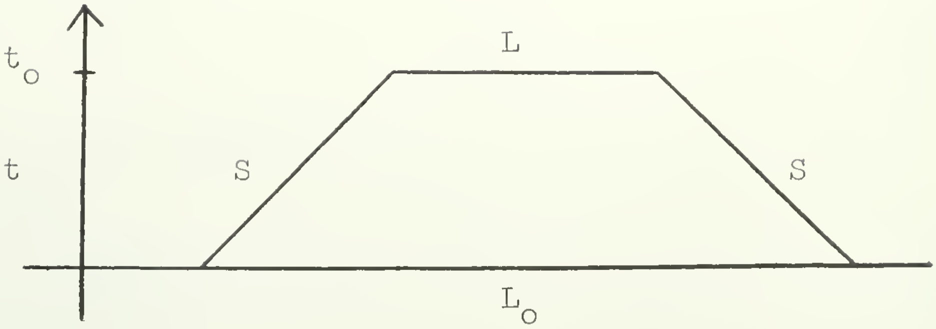

Fig. 5.3

Integration by parts gives

(5.25) $$\int_L (u+v)dx = \int_{L_o} (u+v)dx - \int_S [(u+v)-(f+g)t_x]dx.$$

Since we may add an arbitrary constant to f and g, we can assume that

$$f(0,0) = g(0,0) = 0.$$

Since f and g depend smoothly on u and v,

$$|f| + |g| \leq K(|u|+|v|).$$

Using this estimate and (5.24) we get

$$(5.26) \qquad (u+v) - (f+g)t_x \geq (|u|+|v|)(1-K|t_x|) - 2KM$$

We choose now the sides of the trapezoid so that $1 = K|t_x|$; this makes the length of each side in the x direction $= Kt_o$. Using (5.26) in the second integral on the right in (5.25), and (5.24) on the left we obtain

$$\int_L (|u|+|v|-4KM)dx \leq \int_{L_o} (|u|+|v|)dx + 4KMKt_o.$$

i.e.

$$\int_L (|u|+|v|)dx \leq (4KL + 2Kt_o + 4K^2t_o)M.$$

Since w and z are smooth functions of u, v, and since M is an abbreviation for $\|U(0)\|_\infty$, inequality (5.14) of Lemma 5.1 follows from the above inequality. This completes the proof of Theorem 5.1.

We turn now to the study of periodic solutions:

Theorem 5.2: Suppose that the initial values of U are periodic with respect to x_1 with period p. Then

a) There exists a solution $U(x,t)$ which is periodic with respect to x, with period p, for all t.

b) The average $\overline{U}$ of U over a space period is independent of t.

c) $V(t)$, the total variation of U over a space period satisfies the decay law

$$(5.27) \qquad \frac{V(t)}{p} \leq \frac{H_1}{t} .$$

d) $$|U(x,t) - \overline{U}| \leq \frac{H_1 p}{t} .$$

Proof: It follows from the construction described in Section 2 that if the initial value has space period p, so do all approximate solutions for all t, and therefore so does their limit. This proves a).

To prove b) integrate the conservation law for U over the rectangle $(0,p) \times (0,t)$ and observe that because of periodicity the flux integrals cancel.

To prove c) we take the interval L in inequality (5.1) to have length mp, m so large that $mp \geq t$. The total variation of U over such an L is $mV(t)$, so by (5.1)

$$mV(t) \leq \frac{H_1 mp}{t}$$

which implies (5.27).

Part d) follows from b) and c) since total oscillation is bounded by the total variation per period.

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